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TRANSCENDENTAL DEGREE IN POWER SERIES RINGS

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ABSTRACT. Let D be an integral domain with quotient field K. Let $D[[x]]$ and $K[[x]]$ be the power series ring over D and K, respectively. In this paper, we show that either (1) $K[[x]]$ and $D[[x]]$ have the same quotient field or (2) the quotient field of $K[[x]]$ has uncountable transcendence degree over that of $D[[x]]$, i.e., $tr.d.(K[[x]]/D[[x]]) \geq \aleph_1$. In (2), the bound $\aleph_1$ is the greatest lower bound that one can obtain since under the continuum hypothesis the cardinality of the quotient field of $K[[x]]$ is exactly $\aleph_1$ provided that K is countable. We also show that the above result holds when K is replaced by any quotient overring D_S of D.

1. Introduction

Let D be an integral domain and let $D[x]$ (respectively, $D[[x]]$) be the polynomial ring (respectively, the power series ring) over D. Let K be the quotient field of D. It is easy to see that the polynomial rings $D[x]$ and $K[x]$ have the same quotient field. For the power series rings $D[[x]]$ and $K[[x]]$, Gilmer [4] showed that the following are equivalent.

(1) $D[[x]]$ and $K[[x]]$ have the same quotient field.
(2) $K[[x]] = (D[[x]])_{D^*}$, where $D^* = D \setminus \{0\}$.
(3) If $\{a_i\}_{i=1}^{\infty}$ is a sequence of nonzero elements of D, then $\cap_{i=1}^{\infty} a_i D \neq (0)$.

According to this result, the quotient fields of $D[[x]]$ and $K[[x]]$ are different in general. In fact, as mentioned in [3], it is rare that $D[[x]]$ and $K[[x]]$ have the same quotient field. Except for the trivial case when D is a field, the only example showing that $D[[x]]$ and $K[[x]]$ have the same quotient field is given by Gilmer in [4].

For two integral domains $D_1 \subseteq D_2$, denote by $tr.d.(D_2/D_1)$ the transcendence degree of the quotient field of D_2 over that of D_1. Hence, for a cardinal number α, $tr.d.(D_2/D_1) \geq \alpha$ if there exists a subset of the quotient field of D_2 with cardinality at least α that is algebraically independent over the quotient field of D_1. Suppose that the quotient fields of $K[[x]]$ and $D[[x]]$ are different. Then a natural question is "how large the difference is in this case?" Sheldon showed in [7] that if D contains a nonzero element a such that $\cap_{i=1}^{\infty} a^i D = (0)$, then $tr.d.(D[[x/a]]/D[[x]]) \geq \aleph_0$ and hence $tr.d.(K[[x]]/D[[x]]) \geq \aleph_0$ since $D[[x/a]] \subseteq D[1/a][[x]] \subseteq K[[x]]$. In [1], Arnold and Boyd made a great improvement of this result by showing that if $K[[x]]$ and $D[[x]]$ have different quotient fields, then $tr.d.(K[[x]]/D[[x]]) \geq \aleph_0$.

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