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Projective dimensions and extensions of modules from tilted to cluster-tilted algebras ☆

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We study the module categories of a tilted algebra C and the corresponding cluster-tilted algebra $B = C \ltimes E$ where E is the C-C-bimodule $\mathrm{Ext}^2_C(DC, C)$. We investigate how various properties of a C-module are affected when considered in the module category of B. We give a complete characterization of the projective dimension of a C-module inside mod B. If a C-module M satisfies $\mathrm{Ext}^1_C(M, M) = 0$, we show two sufficient conditions for M to satisfy $\mathrm{Ext}^1_B(M, M) = 0$. In particular, if M_C is indecomposable and $\mathrm{Ext}^1_C(M, M) = 0$, we prove M_B always satisfies $\mathrm{Ext}^1_B(M, M) = 0$.

1. Introduction

We are interested in studying the representation theory of cluster-tilted algebras which are finite dimensional associative algebras that were introduced by Buan, Marsh, and Reiten in [8] and, independently, by Caldero, Chapoton, and Schiffler in [11] for type $\mathbb{A}$.

One motivation for introducing these algebras came from Fomin and Zelevinsky's cluster algebras [13]. Cluster-tilted algebras are the endomorphism algebras of the so-called

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tilting objects in the cluster category of [7]. Many people have studied cluster-tilted algebras in this context, see for example [8–10,12,16].

The second motivation came from classical tilting theory, see [14]. Tilted algebras are the endomorphism algebras of tilting modules over hereditary algebras, whereas cluster-tilted algebras are the endomorphism algebras of cluster-tilting objects over cluster categories of hereditary algebras. This similarity in the two definitions lead to the following precise relation between tilted and cluster-tilted algebras, which was established in [2].

There is a surjective map

$$\{\text{tilted algebras}\} \longmapsto \{\text{cluster-tilted algebras}\}$$
$$C \longmapsto B = C \ltimes E$$

where E denotes the C-C-bimodule $E = \mathrm{Ext}^2_C(DC, C)$ and $C \ltimes E$ is the trivial extension.

This result allows one to define cluster-tilted algebras without using the cluster category. It is natural to ask how the module categories of C and B are related. In this work, we investigate how various properties of a C-module are affected when the same module is viewed as a B-module via the standard embedding. We let M be a right C-module and define a right $B = C \ltimes E$ action on M by

$$M \times B \to M \quad , \quad (m, (c, e)) \mapsto mc.$$

Our first main result is on the projective dimension of a C-module when viewed as a B-module. Here, τ_C^{-1} and Ω_C^{-1} denote respectively the inverse Auslander–Reiten translation and first cosyzygy of a C-module. We note that, because C is tilted, the possible projective dimensions of a module M_C are 0, 1, and 2, and because B is 1-Gorenstein, the possible projective dimensions of a module M_B are 0, 1, and ∞.

Theorem 1.1. *Let C be a tilted algebra, $E = \mathrm{Ext}^2_C(DC, C)$, and $B = C \ltimes E$ the corresponding cluster-tilted algebra. Let M be an indecomposable C-module.*

(a) *If $\mathrm{pd}_C\, M = 0$, then $\mathrm{pd}_B\, M = 0$ if and only if $\mathrm{id}_C\, M \leq 1$. Otherwise, $\mathrm{pd}_B\, M = \infty$.*
(b) *If $\mathrm{pd}_C\, M = 2$, then $\mathrm{pd}_B\, M = \infty$.*
(c) *Let $\mathrm{pd}_C\, M = 1$ with minimal projective resolution $0 \to P_1 \to P_0 \to M \to 0$. Then $\mathrm{pd}_B\, M = 1$ if and only if $\mathrm{id}_C\, M \leq 1$ and $\tau_C^{-1}\Omega_C^{-1}P_0 \cong \tau_C^{-1}\Omega_C^{-1}P_1$. Otherwise, $\mathrm{pd}_B\, M = \infty$.*

Our second main result is on C-modules that satisfy $\mathrm{Ext}^1_C(M, M) = 0$. These are known as *rigid* modules. Here, our result holds when C is a triangular algebra of global dimension equal to 2, and not necessarily tilted. We determine two sufficient conditions to guarantee when a rigid C-module remains rigid when viewed as a B-module, i.e., $\mathrm{Ext}^1_B(M, M) = 0$. Here, τ_C and Ω_C denote respectively the Auslander–Reiten translation and first syzygy of a C-module.