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Bounded Engel and solvable unitary units in group rings ☆

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ABSTRACT

Let FG be the group ring of a group G over a field F. We consider the group of unitary units of FG with respect to the classical involution. Under suitable restrictions upon F, we show that if the unitary units of FG are both bounded Engel and solvable, then the entire unit group of FG is nilpotent. This extends a result of Fisher, Parmenter and Sehgal.

1. Introduction

Let G be a group and F a field of characteristic $p \neq 2$. It is a natural problem to try to determine the structure of the unit group $\mathcal{U}(FG)$ of the group ring FG. In particular, we would like to know the conditions under which $\mathcal{U}(FG)$ satisfies various group identities. This topic has been studied extensively, and we refer to [8] for an overview.

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In particular, it is known when $\mathcal{U}(FG)$ is nilpotent. For modular group rings, Khripta presented the answer in [7]. The case where FG is not modular was handled independently in Fisher–Parmenter–Sehgal [2] and Khripta [6].

Certain subsets of $\mathcal{U}(FG)$ are also interesting. Consider the classical involution on FG, given by $(\sum_{g\in G}\alpha_g g)^* = \sum_{g\in G}\alpha_g g^{-1}$. The identities satisfied by the set of symmetric units (namely, those satisfying $\alpha^* = \alpha$), have also received a good deal of attention and again, we refer to [8] for a discussion.

However, the unitary units are also worthy of study. If R is any ring with involution $*$, then we let $Un(R) = \{r \in \mathcal{U}(R) : rr^* = 1\}$. It is easy to see that $Un(R)$ is a subgroup of $\mathcal{U}(R)$. When $R = FG$, we will always let $*$ be the classical involution. As such, we note that G is a subgroup of $Un(FG)$.

Only a handful of papers have examined identities satisfied by the unitary units. Giambruno–Polcino Milies [3] and Broche–Dooms–Ruiz [1] looked at general results concerning group identities. Gonçalves–Passman [4] discussed when $Un(FG)$ contains a nonabelian free group. Recently, the authors in [10] considered group rings whose unitary units form a nilpotent group; as it turns out, this is usually enough to imply that $\mathcal{U}(FG)$ is nilpotent, but there are exceptions.

However, the Fisher–Parmenter–Sehgal paper did more. Let us establish some notation. On any group G, let $(g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2$ and, recursively,

$$(g_1, \dots, g_{n+1}) = ((g_1, \dots, g_n), g_{n+1}).$$

Then, of course, G is nilpotent if, for some n, we have $(g_1, \dots, g_n) = 1$ for all $g_i \in G$. We say that G is n-Engel if, for every $g, h \in G$, we have

$$(g, \underbrace{h, \dots, h}_{n \text{ times}}) = 1$$

and bounded Engel if it is n-Engel for some n. Also, we let $(g_1, g_2)^o = (g_1, g_2)$ and, recursively,

$$(g_1, \dots, g_{2^{n+1}})^o = ((g_1, \dots, g_{2^n})^o, (g_{2^n+1}, \dots, g_{2^{n+1}})^o).$$

If G is solvable then, for some n, we have $(g_1, \dots, g_{2^n})^o = 1$ for all $g_i \in G$. Of course, every nilpotent group is both bounded Engel and solvable, but even the bounded Engel property and solvability together are not enough to guarantee nilpotence. However, it is shown in [2] that if FG is not modular, then whenever $\mathcal{U}(FG)$ is both bounded Engel and solvable, it is also nilpotent.

Inspired by this result, we ask if it is sufficient to assume that the unitary units are both bounded Engel and solvable, in order to prove that the entire unit group is nilpotent. We show that, under certain restrictions upon the field, this is the case.