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Integral closure and bounds for quotients of multiplicities of monomial ideals

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ABSTRACT

Given a pair of monomial ideals I and J of finite colength of the ring of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$, we prove that some power of I admits a reduction formed by homogeneous polynomials with respect to the Newton filtration induced by J if and only if the quotient of multiplicities $e(I)/e(J)$ attains a suitable upper bound expressed in terms of the Newton polyhedra of I and J. We also explore other connections between mixed multiplicities, Newton filtrations and the integral closure of ideals.

1. Introduction

Let us denote by $\mathcal{O}_n$ the ring of complex analytic function germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a complex analytic map. We say that g is *finite* when $g^{-1}(0) = \{0\}$; in this case, we refer to the number $e(g) = \dim_{\mathbb{C}} \mathcal{O}_n/I(g)$ as the *multiplicity of* g, where $I(g)$ denotes the ideal of $\mathcal{O}_n$ generated by the components of g (see [1, §5], [8, §2] or [9, §2] for several characterizations of this number). More generally, if I is any

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ideal of $\mathcal{O}_n$ of finite colength, then the multiplicity of I, in the sense of Hilbert–Samuel, is denoted by $e(I)$ (see [10,12,23]). We recall that, when I admits a generating system formed by n elements, then $e(I) = \dim_{\mathbb{C}} \mathcal{O}_n/I$. It is well-known that, if we fix a vector $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geq 1}$ and g is semi-weighted homogeneous with respect to w, then $e(g)$ can be expressed as

$$e(g) = \frac{d_1 \cdots d_n}{w_1 \cdots w_n}$$

where d_i is the degree of g_i with respect to w, for all $i = 1, \dots, n$ (see for instance [1, §12.3] or [8, §10.3]). This result was generalized in [7] by replacing the weighted homogeneous filtration induced by w by the Newton filtration induced by a given Newton polyhedron of $\mathbb{R}^n_{\geq 0}$ (see Theorem 4.2). That is, let $\Gamma_+ \subseteq \mathbb{R}^n_{\geq 0}$ be a Newton polyhedron such that $\Gamma_+ \neq \mathbb{R}^n_{\geq 0}$ and Γ_+ intersects each coordinate axis. Let Γ be the union of all compact faces of Γ_+ and let ν_Γ be the Newton filtration induced by Γ_+ (see Section 4 for details). If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is any finite analytic map, then

$$e(g) \geq \frac{d_1 \cdots d_n}{M_\Gamma^n} n! \, \mathrm{V}_n \left(\mathbb{R}^n_{\geq 0} \smallsetminus \Gamma_+ \right), \tag{1}$$

where $d_i = \nu_\Gamma(g_i)$, for all $i = 1, \dots, n$, V_n denotes the n-dimensional volume and M_Γ is the value of ν_Γ over the monomials whose exponent belongs to Γ. The maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ for which equality holds in (1) are called *non-degenerate on* Γ_+. This class of maps is characterized in [7, Theorem 3.3].

If K is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we recall that the multiplicity of K is expressed as $e(K) = n! \, \mathrm{V}_n(\mathbb{R}^n_{\geq 0} \smallsetminus \Gamma_+(K))$, where $\Gamma_+(K)$ denotes the Newton polyhedron of K (see for instance [21,22]). Therefore, relation (1) also shows a lower bound for the quotient $e(g)/e(J)$, where J is the integrally closed monomial ideal such that $\Gamma_+ = \Gamma_+(J)$. We also refer to non-degenerate maps on Γ_+ as *J-non-degenerate maps*. We show that equality holds in (1) if and only if there exists some integers $a_1, \dots, a_n, d \in \mathbb{Z}_{\geq 1}$ such that $\overline{\langle g_1^{a_1}, \dots, g_n^{a_n} \rangle} = \overline{J^d}$, where the bar denotes integral closure.

Moreover, if I is a monomial ideal of $\mathcal{O}_n$ of finite colength, then we use the respective Newton polyhedra of I and J to define an increasing sequence of positive rational numbers $a_{1,J}(I), \dots, a_{n,J}(I)$ that leads to an upper bound for the quotient $e(I)/e(J)$, that is,

$$\frac{e(I)}{e(J)} \leq \frac{a_{1,J}(I) \cdots a_{n,J}(I)}{M_J^n}, \tag{2}$$

where M_J is a positive integer defined in terms of the Newton filtration of J (see Section 4). We prove that equality holds in (2) if and only if there exists some $s \geq 1$ such that $\overline{I^s} = \overline{\langle g_1, \dots, g_n \rangle}$, for some map $(g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ which is J-non-degenerate. This result appears in Theorem 5.5. The proof of this result is preceded