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Koszul almost complete intersections

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ABSTRACT

Let $R = S/I$ be a quotient of a standard graded polynomial ring S by an ideal I generated by quadrics. If R is Koszul, a question of Avramov, Conca, and Iyengar asks whether the Betti numbers of R over S can be bounded above by binomial coefficients on the minimal number of generators of I. Motivated by previous results for Koszul algebras defined by three quadrics, we give a complete classification of the structure of Koszul almost complete intersections and, in the process, give an affirmative answer to the above question for all such rings.

1. Introduction

Let k be a field, S be a standard graded polynomial ring over k, $I \subseteq S$ be a graded ideal, and $R = S/I$. We say that R is a *Koszul algebra* if $k \cong R/R_+$ has a linear free resolution over R. Many rings arising from algebraic geometry are Koszul, including the coordinate rings of Grassmannians [22], sets of $r \leq 2n$ points in general position in $\mathbb{P}^n$ [23], and canonical embeddings of smooth curves under mild restrictions [25], as well as all suitably high Veronese subrings of any standard graded algebra [5]. However, the simplest examples of Koszul algebras, due to Fröberg [19], are quotients by quadratic monomial ideals, and a guiding heuristic in the study of Koszul algebras has been that any reasonable property of algebras defined by quadratic monomial ideals should also

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hold for Koszul algebras; for example, see [3], [15], [4]. Among such properties, considering the Taylor resolution for an algebra defined by a quadratic monomial ideal leads to the following question about the Betti numbers of a Koszul algebra.

Question 1.1 *([3, 6.5]).* If R is Koszul and I is minimally generated by g elements, does the following inequality hold for all i?

$$\beta_i^S(R) \leq \binom{g}{i}$$

In particular, is $\operatorname{pd}_S R \leq g$?

The above questions are known to have affirmative answers when R is LG-quadratic (see next section) and for arbitrary Koszul algebras when $g \leq 3$ by [7, 4.5]. Recall that R or I is called an *almost complete intersection* if I is minimally generated by $\operatorname{ht} I + 1$ elements. The motivation for studying Koszul almost complete intersections comes from the fact that the above question is easily seen to have an affirmative answer when I is a complete intersection or has height one so that the interesting case for Koszul algebras defined by three quadrics is precisely when I is an almost complete intersection. Our main results (Theorem 3.1, Theorem 3.3, Corollary 3.4, Theorem 4.3) show that Question 1.1 has an affirmative answer for Koszul almost complete intersections generated by any number of quadrics; they are summarized in the theorem below.

Main Theorem. *Let $R = S/I$ be a Koszul almost complete intersection with I minimally generated by $g + 1$ quadrics for some $g \geq 1$. Then $\beta_{2,3}^S(R) \leq 2$, and:*

(a) *If $\beta_{2,3}^S(R) = 1$, there are linear forms x, z, and w such that $I = (xz, zw, q_3, \dots, q_{g+1})$ for some regular sequence of quadrics $q_3, \dots, q_{g+1}$ on $S/(xz, zw)$.*
(b) *If $\beta_{2,3}^S(R) = 2$, there is a 3×2 matrix of linear forms M with $\operatorname{ht} I_2(M) = 2$ such that $I = I_2(M) + (q_4, \dots, q_{g+1})$ for some regular sequence of quadrics $q_4, \dots, q_{g+1}$ on $S/I_2(M)$.*

Furthermore, R is LG-quadratic and, therefore, satisfies $\beta_i^S(R) \leq \binom{g+1}{i}$ for all i.

The division of the rest of the paper is as follows. We recount various properties and examples of Koszul algebras and their Betti tables in §2 which will be important in the sequel. In §3, we determine the structure of Koszul almost complete intersections with either one or two linear second syzygies. We then complete the classification of Koszul almost complete intersections in §4 by showing that every quadratic almost complete intersection has at most two linear second syzygies.

Notation. Throughout the remainder of the paper, the following notation will be in force unless specifically stated otherwise. Let k be a fixed ground field of arbitrary