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## Journal of Algebra

www.elsevier.com/locate/jalgebra

# On the completion of the mapping class group of genus two



ALGEBRA

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#### A R T I C L E I N F O

Article history: Received 11 January 2017 Available online 11 January 2018 Communicated by Vera Serganova

#### Keywords: Moduli space of curves Moduli space of principally polarized abelian varieties Mapping class groups Unipotent completion Hodge Lie algebra Presentation of nilpotent Lie algebras Classical modular forms Special values of L-functions

#### ABSTRACT

In this paper, we will study the Lie algebra of the prounipotent radical of the relative completion of the mapping class group of genus two. In particular, we will partially determine a minimal presentation of the Lie algebra by determining the generators and bounding the degree of the relations of it.

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### 1. Introduction

Let X be a complex smooth algebraic variety. Fix a point x in X. The Lie algebra  $\mathfrak{g}$  of the unipotent (Malcev) completion over  $\mathbb{Q}$  of  $\pi_1(X, x)$  carries a natural  $\mathbb{Q}$ -MHS. A minimal presentation of  $\operatorname{Gr}^W_{\bullet} \mathfrak{g}$  was studied by Morgan in [14]. It is generated in weight -1 and -2 by  $H_1(X, \mathbb{Q})$  and has relations in weight -2, -3, and -4 coming from  $H_2(X, \mathbb{Q})$ . Denote by  $\mathcal{M}_q$  the moduli stack of smooth projective curves of genus g over  $\mathbb{C}$ .

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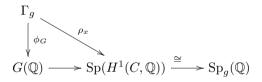
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Assume that  $g \geq 2$ . It will be considered as an orbifold in this paper. It is a natural problem to look for a presentation of the Lie algebra of the unipotent completion of the orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{M}_g, x)$  of  $\mathcal{M}_g$ , since  $\mathcal{M}_g$  is smooth and has a finite cover by a smooth algebraic variety. The issue is that the rational homology  $H_1(\mathcal{M}_g, \mathbb{Q})$ vanishes and hence the unipotent completion is trivial. Instead, we will consider the relative completion of a discrete group due to Deligne, which generalizes the unipotent completion of a group. Associated to the universal family  $f : \mathcal{C}_g \to \mathcal{M}_g$ , there is a natural monodromy representation in the orbifold sense

$$\rho_x: \pi_1^{\operatorname{orb}}(\mathcal{M}_g, x) \to \operatorname{Sp}(H^1(C, \mathbb{Q})),$$

where C is the fiber of f over x. The orbifold fundamental group  $\pi_1^{\text{orb}}(\mathcal{M}_g, x)$  is naturally isomorphic to the mapping class group  $\Gamma_g$ . It is the group of isotopy classes of orientation-preserving diffeomorphisms of a compact oriented surface  $S_g$  of genus g. The relative completion  $\mathcal{G}_g$  of  $\Gamma_g$  with respect to  $\rho_x$  is the inverse limit of all algebraic group G over  $\mathbb{Q}$  that is an extension of  $\operatorname{Sp}_g$  by a unipotent group over  $\mathbb{Q}$  and such that there is a Zariski-dense representation  $\phi_G : \Gamma_g \to G(\mathbb{Q})$  making the diagram



commute, where the isomorphism  $\operatorname{Sp}(H^1(C, \mathbb{Q})) \cong \operatorname{Sp}_g(\mathbb{Q})$  is given by fixing a symplectic basis for  $H^1(C, \mathbb{Q})$ . The completion  $\mathcal{G}_g$  is a proalgebraic group over  $\mathbb{Q}$  that is an extension of  $\operatorname{Sp}_g$  with a prounipotent group  $\mathcal{U}_g$  over  $\mathbb{Q}$ . In [8], Hain further developed the theory of relative completion and constructed a canonical MHS on the Lie algebra  $\mathfrak{u}_g$  of  $\mathcal{U}_g$ . In this paper, we will study the associated graded Lie algebra  $\operatorname{Gr}_{\bullet}^W \mathfrak{u}_g$  for g = 2.

Hain proved in [6,9] that for  $g \ge 4$ , the Lie algebra  $\mathfrak{u}_g$  is generated in weight -1 and quadratically presented and that for g = 3, it is generated in weight -1 and admits quadratic and cubic relations. Furthermore, Hain used this result to prove that the Lie algebra  $\mathfrak{t}_g$  of the unipotent completion of the Torelli group  $T_g$  for  $g \ge 3$  is finitely presented with quadratic and possible cubic relations. This gave an important insight on the open problem of the finite presentation of  $T_g$  for  $g \ge 3$ . It is known that  $T_g$  is finitely generated for  $g \ge 3$ , but for any  $g \ge 3$ , it remains open whether or not  $T_g$  is finitely presented. For the case when g = 2, Mess showed in [13] that  $T_2$  is countably generated free group. Therefore, the Lie algebra  $\mathfrak{t}_2$  is simply the inverse limit of all nilpotent quotients of the free Lie algebra  $\mathfrak{t}_2$  and  $\mathfrak{u}_2$  are essentially different, while for  $g \ge 3$ ,  $\mathfrak{t}_g$  is the central extension of  $\mathfrak{u}_g$  by  $\mathbb{G}_a$  (see [5]).

Hain used the Johnson's fundamental work [3] to determine the generators and Kabanov's theorem [11] on  $H^2(\mathcal{M}_g, \mathbb{V})$ , where  $\mathbb{V}$  is a symplectic local system, to bound Download English Version:

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