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On the completion of the mapping class group of genus two

Tatsunari Watanabe

Department of Mathematics, Purdue University, West Lafayette, IN, United States

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ABSTRACT

In this paper, we will study the Lie algebra of the prounipotent radical of the relative completion of the mapping class group of genus two. In particular, we will partially determine a minimal presentation of the Lie algebra by determining the generators and bounding the degree of the relations of it.

1. Introduction

Let X be a complex smooth algebraic variety. Fix a point x in X. The Lie algebra $\mathfrak{g}$ of the unipotent (Malcev) completion over $\mathbb{Q}$ of $\pi_1(X, x)$ carries a natural $\mathbb{Q}$-MHS. A minimal presentation of $\mathrm{Gr}^W_\bullet \mathfrak{g}$ was studied by Morgan in [14]. It is generated in weight -1 and -2 by $H_1(X, \mathbb{Q})$ and has relations in weight -2, -3, and -4 coming from $H_2(X, \mathbb{Q})$. Denote by $\mathcal{M}_g$ the moduli stack of smooth projective curves of genus g over $\mathbb{C}$.

E-mail address: twatana@purdue.edu.
URL: http://www.math.purdue.edu/people/bio/twatana.

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Assume that $g \geq 2$. It will be considered as an orbifold in this paper. It is a natural problem to look for a presentation of the Lie algebra of the unipotent completion of the orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{M}_g, x)$ of $\mathcal{M}_g$, since $\mathcal{M}_g$ is smooth and has a finite cover by a smooth algebraic variety. The issue is that the rational homology $H_1(\mathcal{M}_g, \mathbb{Q})$ vanishes and hence the unipotent completion is trivial. Instead, we will consider the relative completion of a discrete group due to Deligne, which generalizes the unipotent completion of a group. Associated to the universal family $f : \mathcal{C}_g \to \mathcal{M}_g$, there is a natural monodromy representation in the orbifold sense

$$\rho_x : \pi_1^{\mathrm{orb}}(\mathcal{M}_g, x) \to \mathrm{Sp}(H^1(C, \mathbb{Q})),$$

where C is the fiber of f over x. The orbifold fundamental group $\pi_1^{\mathrm{orb}}(\mathcal{M}_g, x)$ is naturally isomorphic to the mapping class group Γ_g. It is the group of isotopy classes of orientation-preserving diffeomorphisms of a compact oriented surface S_g of genus g. The relative completion $\mathcal{G}_g$ of Γ_g with respect to ρ_x is the inverse limit of all algebraic group G over $\mathbb{Q}$ that is an extension of Sp_g by a unipotent group over $\mathbb{Q}$ and such that there is a Zariski-dense representation $\phi_G : \Gamma_g \to G(\mathbb{Q})$ making the diagram

$$\begin{array}{ccccc} \Gamma_g & & & & \\ \downarrow{\scriptstyle \phi_G} & \searrow^{\rho_x} & & & \\ G(\mathbb{Q}) & \longrightarrow & \mathrm{Sp}(H^1(C, \mathbb{Q})) & \xrightarrow{\cong} & \mathrm{Sp}_g(\mathbb{Q}) \end{array}$$

commute, where the isomorphism $\mathrm{Sp}(H^1(C, \mathbb{Q})) \cong \mathrm{Sp}_g(\mathbb{Q})$ is given by fixing a symplectic basis for $H^1(C, \mathbb{Q})$. The completion $\mathcal{G}_g$ is a proalgebraic group over $\mathbb{Q}$ that is an extension of Sp_g with a prounipotent group $\mathcal{U}_g$ over $\mathbb{Q}$. In [8], Hain further developed the theory of relative completion and constructed a canonical MHS on the Lie algebra $\mathfrak{u}_g$ of $\mathcal{U}_g$. In this paper, we will study the associated graded Lie algebra $\mathrm{Gr}^W_\bullet\, \mathfrak{u}_g$ for $g = 2$.

Hain proved in [6,9] that for $g \geq 4$, the Lie algebra $\mathfrak{u}_g$ is generated in weight -1 and quadratically presented and that for $g = 3$, it is generated in weight -1 and admits quadratic and cubic relations. Furthermore, Hain used this result to prove that the Lie algebra $\mathfrak{t}_g$ of the unipotent completion of the Torelli group T_g for $g \geq 3$ is finitely presented with quadratic and possible cubic relations. This gave an important insight on the open problem of the finite presentation of T_g for $g \geq 3$. It is known that T_g is finitely generated for $g \geq 3$, but for any $g \geq 3$, it remains open whether or not T_g is finitely presented. For the case when $g = 2$, Mess showed in [13] that T_2 is countably generated free group. Therefore, the Lie algebra $\mathfrak{t}_2$ is simply the inverse limit of all nilpotent quotients of the free Lie algebra generated by $H_1(T_2, \mathbb{Q})$. However, our main result suggests that the Lie algebras $\mathfrak{t}_2$ and $\mathfrak{u}_2$ are essentially different, while for $g \geq 3$, $\mathfrak{t}_g$ is the central extension of $\mathfrak{u}_g$ by $\mathbb{G}_a$ (see [5]).

Hain used the Johnson's fundamental work [3] to determine the generators and Kabanov's theorem [11] on $H^2(\mathcal{M}_g, \mathbb{V})$, where $\mathbb{V}$ is a symplectic local system, to bound