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ON WEAK EQUIVALENCES OF GRADINGS

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Abstract. When one studies the structure (e.g. graded ideals, graded subspaces, radicals,...) or graded polynomial identities of graded algebras, the grading group itself does not play an important role, but can be replaced by any other group that realizes the same grading. Here we come to the notion of weak equivalence of gradings: two gradings are weakly equivalent if there exists an isomorphism between the graded algebras that maps each graded component onto a graded component. The following question arises naturally: when a group grading on a finite dimensional algebra is weakly equivalent to a grading by a finite group? It turns out that this question can be reformulated purely group theoretically in terms of the universal group of the grading. Namely, a grading is weakly equivalent to a grading by a finite group if and only if the universal group of the grading is residually finite with respect to a special subset of the grading group. The same is true for all the coarsenings of the grading if and only if the universal group of the grading is hereditarilly residually finite with respect to the same subset. We show that if $n \geqslant 349$, then on the full matrix algebra $M_n(F)$ there exists an elementary group grading that is not weakly equivalent to any grading by a finite (semi)group, and if $n \leqslant 3$, then any elementary grading on $M_n(F)$ is weakly equivalent to an elementary grading by a finite group.

1. Introduction

When studying graded algebras, one has to determine, when two graded algebras are considered "the same" or equivalent.

Recall that a decomposition $\Gamma\colon A = \bigoplus_{s\in S} A^{(s)}$ of an algebra A over a field F into a direct sum of subspaces $A^{(s)}$ is a *grading* on A by a (semi)group S if $A^{(s)}A^{(t)} \subseteq A^{(st)}$ for all $s, t \in S$. Then we say that S is the *grading (semi)group* of Γ and the algebra A is *graded* by S.

Let

$$\Gamma_1\colon A = \bigoplus_{s\in S} A^{(s)}, \qquad \Gamma_2\colon B = \bigoplus_{t\in T} B^{(t)} \tag{1.1}$$

be two gradings where S and T are (semi)groups and A and B are algebras.

The most restrictive case is when we require that both grading (semi)groups coincide:

Definition 1.1 (e.g. [11, Definition 1.15]). The gradings (1.1) are *isomorphic* if $S = T$ and there exists an isomorphism $\varphi\colon A \xrightarrow{\sim} B$ of algebras such that $\varphi(A^{(s)}) = B^{(s)}$ for all $s \in S$. In this case we say that A and B are *graded isomorphic*.

In some cases, such as in [16], less restrictive requirements are more suitable.

Definition 1.2 ([16, Definition 2.3]). The gradings (1.1) are *equivalent* if there exists an isomorphism $\varphi\colon A \xrightarrow{\sim} B$ of algebras and an isomorphism $\psi\colon S \xrightarrow{\sim} T$ of (semi)groups such that $\varphi(A^{(s)}) = B^{\left(\psi(s)\right)}$ for all $s \in S$.

Remark 1.3. The notion of graded equivalence was considered by Yu. A. Bahturin, S. K. Seghal, and M. V. Zaicev in [6, Remark after Definition 3]. In the paper of V. Mazorchuk and K. Zhao [20] it appears under the name of graded isomorphism. A. Elduque

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