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Identities in upper triangular tropical matrix semigroups and the bicyclic monoid

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ABSTRACT

We establish necessary and sufficient conditions for a semigroup identity to hold in the monoid of $n \times n$ upper triangular tropical matrices, in terms of equivalence of certain tropical polynomials. This leads to an algorithm for checking whether such an identity holds, in time polynomial in the length of the identity and size of the alphabet. It also allows us to answer a question of Izhakian and Margolis, by showing that the identities which hold in the monoid of 2×2 upper triangular tropical matrices are exactly the same as those which hold in the bicyclic monoid. Our results extend to a broader class of "chain structured tropical matrix semigroups"; we exhibit a faithful representation of the free monogenic inverse semigroup within such a semigroup, which leads also to a representation by 3×3 upper triangular tropical matrices.

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1. Introduction

Over the last few years there has been considerable interest in the structure of the monoid $M_n(\mathbb{T})$ of tropical (max-plus) matrices under multiplication, with lines of investigation including characterisations of Green's relations [12,15,24,25], structural properties of subsemigroups and maximal subgroups [8,11,22], as well as connections between the algebraic properties of elements (or indeed subsemigroups) and their actions upon tropically convex sets [21,27,38].

A natural question, especially in view of d'Alessandro and Pasku's proof that finitely generated subsemigroups of $M_n(\mathbb{T})$ have polynomial growth [8], is whether $M_n(\mathbb{T})$ satisfies a nontrivial semigroup identity. Izhakian and Margolis [23] were the first to consider this question, in the 2×2 case; they showed that $M_2(\mathbb{T})$ does indeed satisfy a nontrivial identity. A key step of their proof was establishing an identity for the upper triangular submonoid $UT_2(\mathbb{T})$, which is also of interest in its own right. Specifically, they showed that $UT_2(\mathbb{T})$ satisfies the celebrated identity $AB^2A^2BAB^2A = AB^2ABA^2B^2A$, which was shown by Adjan [1] to hold in the bicyclic monoid $\mathcal{B}$. Since $\mathcal{B}$ embeds in $UT_2(\mathbb{T})$, every identity satisfied in the latter must also hold in the former. In view of this and their results, Izhakian and Margolis posed the natural question of whether the converse holds, that is, whether $UT_2(\mathbb{T})$ and $\mathcal{B}$ satisfy exactly the same identities, or equivalently (by Birkhoff's HSP theorem [4]), whether they generate the same variety.

In Section 3 we develop an exact characterisation of identities which hold in $UT_2(\mathbb{T})$, by associating k tropical polynomial equations in k variables to each identity on k letters. This gives a simple and algorithmically efficient (as we shall see in Section 8) method to check whether any given identity holds. In Section 4, by considering the embedding of $\mathcal{B}$ into $UT_2(\mathbb{T})$, we use this to show that an identity which fails to hold in $UT_2(\mathbb{T})$ must also fail in $\mathcal{B}$, thus answering the above-mentioned question of Izhakian and Margolis [23] and establishing that the bicyclic monoid $\mathcal{B}$ and the upper triangular tropical matrix monoid $UT_2(\mathbb{T})$ satisfy exactly the same semigroup identities (Theorem 4.1).

It follows by Birkhoff's HSP Theorem, of course, that $\mathcal{B}$ and $UT_2(\mathbb{T})$ generate the same variety, and hence share all properties of monoids which are visible in the variety generated. For example, by a theorem of Shneerson [36], $\mathcal{B}$ has infinite axiomatic rank, so $UT_2(\mathbb{T})$ must also have. In particular, since infinite axiomatic rank implies the nonexistence of a finite basis, from our theorem plus [36] we can deduce the recent result of Chen, Hu, Luo and Sapir [7] that the variety generated by $UT_2(\mathbb{T})$ shares with that generated by $\mathcal{B}$ the property of not admitting a finite basis of identities.

There is an interesting relationship between these results and work of Pastijn [30], which studies identities in the bicyclic monoid by reduction to checking linear inequalities or, equivalently, properties of certain polyhedral complexes. Once we have established that identities in $\mathcal{B}$ are equivalent to identities in $UT_2(\mathbb{T})$, it becomes evident that our methods (for identities in $UT_2(\mathbb{T})$) are related to Pastijn's (for identities in $\mathcal{B}$). However, since our methods are used to establish the correlation in the first place, we cannot hope to deduce our results from Pastijn's work.