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The rotating normal form of braids is regular

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ABSTRACT

Defined on Birman–Ko–Lee monoids, the rotating normal form has strong connections with the Dehornoy's braid ordering. It can be seen as a process for selecting between all the representative words of a Birman–Ko–Lee braid a particular one, called *rotating* word. In this paper we construct, for all $n \geqslant 2$, a finite-state automaton which recognizes rotating words on n strands, proving that the rotating normal form is regular. As a consequence we obtain the regularity of a σ-definite normal form defined on the whole braid group.

1. Introduction

Originally, the group B_n of n-strand braids was defined as the group of isotopy classes of n-strand geometric braids. An algebraic presentation of B_n was given by E. Artin in [1]:

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{rcll} \sigma_i\sigma_j & = & \sigma_j\sigma_i & \text{for } |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i & = & \sigma_j\sigma_i\sigma_j & \text{for } |i-j| = 1 \end{array} \right\rangle. \tag{1}$$

An n-strand braid is an equivalence class consisting of (infinitely many) words in the letters $\sigma_i^{\pm 1}$. The standard correspondence between elements of the presented group B_n

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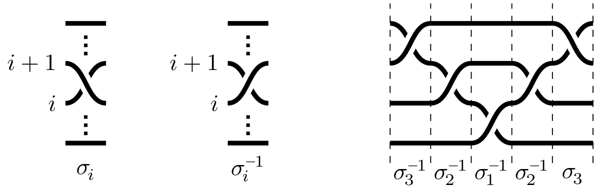

Fig. 1. Interpretation of a word in the letters $\sigma_i^{\pm 1}$ as a geometric braid diagram.

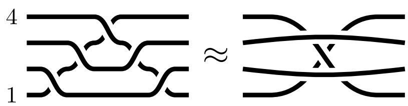

Fig. 2. In the geometric braid $a_{1,4}$, the strands 1 and 4 cross under the strands 2 and 3.

and geometric braids consists in using σ_i as a code for the geometric braid where only the ith and the $(i+1)$st strands cross, with the strand originally at position $(i+1)$ in front of the other (see Fig. 1).

In 1998, J.S. Birman, K.H. Ko, and S.J. Lee [3] introduced and investigated for each n a submonoid B_n^{+*} of B_n, which is known as the *Birman–Ko–Lee* monoid. The name *dual braid monoid* was subsequently proposed because several numerical parameters obtain symmetric values when they are evaluated on the positive braid monoid B_n^+ and on B_n^{+*}, a correspondence that was extended to the more general context of Artin–Tits groups by D. Bessis [2] in 2003. The dual braid monoid B_n^{+*} is the submonoid of B_n generated by the braids $a_{i,j}$ with $1 \leqslant i < j \leqslant n$, where $a_{i,j}$ is defined by $a_{i,j} = \sigma_i \cdots \sigma_{j-1}\, \sigma_j\, \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}$. In geometrical terms, the braid $a_{i,j}$ corresponds to a crossing of the ith and jth strands, both passing behind the (possible) intermediate strands (see Fig. 2).

Remark. In [3], the braid $a_{i,j}$ is defined to be $\sigma_{j-1} \cdots \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$, corresponding to a crossing of the ith and jth strands, both passing in **front** of the (possible) intermediate strands. The two definitions lead to isomorphic monoids. Our choice is this of [13] and has connections with Dehornoy's braid ordering: B_{n-1}^{+*} is an initial segment of B_n^{+*}.

By definition, σ_i equals $a_{i,i+1}$ and, therefore, the positive braid monoid B_n^+ is included in the monoid B_n^{+*}, a proper inclusion for $n \geqslant 3$ since the braid $a_{1,3}$ does not belong to the monoid B_3^+.

For $n \geqslant 2$, we denote by A_n the set $\{a_{p,q} \,|\, 1 \leqslant p < q \leqslant n\}$. If p and q are two integers of $\mathbb{N}$ satisfying $p \leqslant q$, we denote by $[p,q]$ the interval $\{p, ..., q\}$ of $\mathbb{N}$. The interval $[p,q]$ is said to be *nested* in the interval $[r,s]$ if the relation $r < p < q < s$ holds. The following presentation of the monoid B_n^{+*} is given in [3].

Proposition 1.1. *The monoid B_n^{+*} is presented by generators A_n and relations:*

$$a_{p,q}a_{r,s} = a_{r,s}a_{p,q} \quad \textit{for } [p,q] \textit{ and } [r,s] \textit{ disjoint or nested,} \tag{2}$$