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Simultaneous computation of Hecke operators

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A B S T R A C T

We present a new method for computing Hecke operators acting on spaces of algebraic modular forms based on an idea of Eichler's. The method significantly improves the complexity of computing operators that can be obtained this way and in addition computes two operators simultaneously. We show that in certain cases the approach can be used to attain the action of the full Hecke algebra with respect to a hyperspecial subgroup and use it to compute Hecke eigenforms for compact forms of symplectic groups.

1. Introduction

A classical task in the arithmetic theory of quadratic forms is the enumeration of a system of representatives of the isometry classes in a genus. That means given a quadratic space (V, q) over $\mathbb{Q}$ and a $\mathbb{Z}$-lattice $L \subset V$ decompose the set of all lattices that are locally isometric to L at every prime p into (global) isometry classes. The number of isometry classes is known to be finite (even completely known without computation if q is indefinite and the rank is at least 3) and the task is usually settled by use of Kneser's p-neighbor method (cf. [14]).

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Let us now replace the orthogonal group (with respect to q) with another reductive linear algebraic group $\mathbb{G}$ over $\mathbb{Q}$. In this situation we can still ask the same question. Given a faithful representation $\mathbb{G} \hookrightarrow \mathrm{GL}_n$ and a lattice $L \subset \mathbb{Q}^n$ we would like to decompose the $\mathbb{G}(\hat{\mathbb{Q}})$-orbit of L (called the $\mathbb{G}$-genus) into $\mathbb{G}(\mathbb{Q})$-orbits (called $\mathbb{G}$-classes), where $\hat{\mathbb{Q}}$ denotes the finite adeles of $\mathbb{Q}$. Again this question is well-studied, for example it is known that the class number is one, if $\mathbb{G}$ is simply connected, absolutely simple and $\mathbb{G}(\mathbb{R})$ is not compact by virtue of the strong approximation property. On the other hand if $\mathbb{G}(\mathbb{R})$ is compact there are analogues of Kneser's neighbor method that allow us to tackle this problem algorithmically (cf. [10] and the series [4,5,12]). A helpful tool in the latter case is provided by so-called mass formulae. Since $\mathbb{G}(\mathbb{R})$ is assumed to be compact, the stabilizer of a lattice in $\mathbb{G}(\mathbb{Q})$ is finite and hence so is the quantity

$$\mathrm{mass}(L) = \mathrm{mass}(\mathrm{genus}(L)) = \sum_M \frac{1}{|\,\mathrm{Stab}_{\mathbb{G}(\mathbb{Q})}(M)|} \tag{1}$$

where the sum runs over a system of representatives of the $\mathbb{G}$-classes in the $\mathbb{G}$-genus of L. It turns out that the mass only depends on local information on L and is (as long as $\mathbb{G}$ and L are suitably well-behaved) computable without actually writing down a system of representatives. The probably best-known instance of this principle is the Smith–Minkowski–Siegel mass formula in the case of the orthogonal groups ([19]) which was later generalized to the case of arbitrary classical groups ([9]). These mass formulae often a priori only compute the mass of quite restrictive genera (e.g. even unimodular lattices in the case of orthogonal group), hence it is desirable to be able to compare the masses of different genera. The idea how to do this goes back to the work of Eichler (cf. [8]) and works as follows: Let L, L' be two lattices in faithful $\mathbb{G}$-modules and let K, K' be their respective stabilizers in $\mathbb{G}(\hat{\mathbb{Q}})$, then

$$\mathrm{mass}(L)[K : (K \cap K')] = \mathrm{mass}(L')[K' : (K \cap K')]. \tag{2}$$

Moreover the idea behind this can be used to actually compute representatives for certain genera starting from representatives of another genus (cf. [2]).

In this article we apply Eichler's method to the computation of Hecke operators acting on algebraic modular forms as introduced by Gross ([11]). Let $\mathbb{G}$ be a connected, reductive linear algebraic group over $\mathbb{Q}$[1] such that $\mathbb{G}(\mathbb{R})$ is compact and choose an open, compact subgroup $K \subset \mathbb{G}(\hat{\mathbb{Q}})$ as well as an irreducible $\mathbb{Q}$-rational representation V of $\mathbb{G}$. In this notation the space of algebraic modular forms of level K and weight V is the space

$$M(V, K) = \left\{ f : \mathbb{G}(\hat{k}) \to V \mid {}^{f(g\gamma\kappa) = gf(\gamma) \text{ for } \gamma \in \mathbb{G}(\hat{k}),}_{g \in \mathbb{G}(k), \kappa \in K} \right\}. \tag{3}$$

[1] We only limit ourselves to the rationals for the purpose of this introduction while later working over arbitrary (totally real) number fields.