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The congruence subgroup problem for low rank free and free metabelian groups

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To Efim Zelmanov, a friend and a leader

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ABSTRACT

The congruence subgroup problem for a finitely generated group Γ asks whether $\widehat{Aut(\Gamma)} \to Aut(\hat{\Gamma})$ is injective, or more generally, what is its kernel $C(\Gamma)$? Here $\hat{X}$ denotes the profinite completion of X.
In this paper we first give two new short proofs of two known results (for $\Gamma = F_2$ and Φ_2) and a new result for $\Gamma = \Phi_3$:

(1) $C(F_2) = \{e\}$ when F_2 is the free group on two generators.
(2) $C(\Phi_2) = \hat{F}_\omega$ when Φ_n is the free metabelian group on n generators, and $\hat{F}_\omega$ is the free profinite group on $\aleph_0$ generators.
(3) $C(\Phi_3)$ contains $\hat{F}_\omega$.

Results (2) and (3) should be contrasted with an upcoming result of the first author showing that $C(\Phi_n)$ is abelian for $n \geq 4$.

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1. Introduction

The classical congruence subgroup problem (CSP) asks for, say, $G = SL_n(\mathbb{Z})$ or $G = GL_n(\mathbb{Z})$, whether every finite index subgroup of G contains a principal congruence subgroup, i.e. a subgroup of the form $G(m) = \ker(G \to GL_n(\mathbb{Z}/m\mathbb{Z}))$ for some $0 \neq m \in \mathbb{Z}$. Equivalently, it asks whether the natural map $\hat{G} \to GL_n(\hat{\mathbb{Z}})$ is injective, where $\hat{G}$ and $\hat{\mathbb{Z}}$ are the profinite completions of the group G and the ring $\mathbb{Z}$, respectively. More generally, the CSP asks what is the kernel of this map. It is a classical 19th century result that the answer is negative for $n = 2$. Moreover (but not so classical, cf. [20,15]), the kernel, in this case, is $\hat{F}_\omega$ – the free profinite group on a countable number of generators. On the other hand, for $n \geq 3$, the map is injective and the kernel is therefore trivial.

The CSP can be generalized as follows: Let Γ be a group and M a finite index characteristic subgroup of it. Denote:

$$G(M) = \ker(Aut(\Gamma) \to Aut(\Gamma/M)).$$

Such a finite index normal subgroup of $G = Aut(\Gamma)$ will be called a "principal congruence subgroup" and a finite index subgroup of G which contains such a $G(M)$ for some M will be called a "congruence subgroup". Now, the CSP for Γ asks whether every finite index subgroup of G is a congruence subgroup. When Γ is finitely generated, $Aut(\hat{\Gamma})$ is profinite and the CSP is equivalent to the question (cf. [8], §1 and §3): Is the map $\hat{G} = \widehat{Aut(\Gamma)} \to Aut(\hat{\Gamma})$ injective? More generally, it asks what is the kernel $C(\Gamma)$ of this map.

As $GL_n(\mathbb{Z}) = Aut(\mathbb{Z}^n)$, the classical congruence subgroup results mentioned above can therefore be reformulated as $C(A_2) = \hat{F}_\omega$ while $C(A_n) = \{e\}$ for $n \geq 3$, when $A_n = \mathbb{Z}^n$ is the free abelian group on n generators.

Very few results are known when Γ is non-abelian. A very surprising result was proved in [2] by Asada by methods of algebraic geometry:

Theorem 1.1. *$C(F_2) = \{e\}$, i.e., the free group on two generators has the congruence subgroup property, namely $\widehat{Aut(F_2)} \to Aut(\hat{F}_2)$ is injective.*

A purely group theoretic proof for this theorem was given by Bux–Ershov–Rapinchuk [8]. Our first goal in this paper is to give an easier and more direct proof of Theorem 1.1, which also give a better quantitative estimate: we give an explicitly constructed congruence subgroup $G(M)$ of $Aut(F_2)$ which is contained in a given finite index subgroup H of $Aut(F_2)$ of index n. Our estimates on the index of M in F_2 as a function of n are substantially better than those of [8] – see Theorems 2.7 and 2.9.

We then turn to $\Gamma = \Phi_2$, the free metabelian group on two generators. The initial treatment of Φ_2 is similar to F_2, but quite surprisingly, the first named author showed in [4] a negative answer, i.e. $C(\Phi_2) \neq \{e\}$. We also give a shorter proof of this result, deducing that:

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