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Rules of Three for commutation relations[☆]

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Dedicated to Efim Zelmanov on his 60th birthday

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A B S T R A C T

We study the phenomenon in which commutation relations for sequences of elements in a ring are implied by similar relations for subsequences involving at most three indices at a time.

What I tell you three times is true.

The Hunting of the Snark, by Lewis Carroll

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Introduction

In this paper, we investigate the following surprisingly widespread phenomenon which we call *The Rule of Three*: in order for a particular kind of commutation relation to hold for subsequences of elements of a ring labeled by any subset of indices, it is enough that these relations hold for subsets of size one, two, and three.

Here is a typical "Rule of Three" statement. Let $g_1, \ldots, g_N, h_1, \ldots, h_N$ be invertible elements in an associative ring. Then the following are equivalent (cf. Theorem 3.4):

- for any subsequence of indices $1 \le s_1 < \cdots < s_m \le N$, the element $g_{s_m} \cdots g_{s_1}$ commutes with both $h_{s_m} \cdots h_{s_1}$ and $h_{s_m} + \cdots + h_{s_1}$;
- the above condition holds for all subsequences of length $m \le 3$.

We establish many results of this form, including

- Rules of Three for noncommutative elementary symmetric functions (Section 1);
- Rules of Three for generating functions over rings (Section 2);
- Rules of Three for sums and products (Section 3).

Proofs are given in Sections 4–8. For reference, Theorems 2.5 and 2.12 are proved in Section 5; Theorem 3.2 is proved in Section 6; Theorems 1.1, 3.4, 3.5, 3.10, and 3.11 are proved in Section 7; and Theorem 1.6 is proved in Section 8.

1. Rules of Three for noncommutative symmetric functions

Let $\mathbf{u} = (u_1, \ldots, u_N)$ be an ordered N-tuple of elements in a ring R. (We informally view $u_1, \ldots, u_N$ as "noncommuting variables.") For an integer k, the *noncommutative elementary symmetric function* $e_k(\mathbf{u}) \in R$ is defined by

$$e_k(\mathbf{u}) = \sum_{N \ge i_1 > i_2 > \cdots > i_k \ge 1} u_{i_1} u_{i_2} \cdots u_{i_k} \,. \tag{1.1}$$

(By convention, $e_0(\mathbf{u}) = 1$ and $e_k(\mathbf{u}) = 0$ if $k < 0$ or $k > N$.) More generally, for a subset $S \subset \{1, \ldots, N\}$, we denote

$$e_k(\mathbf{u}_S) = \sum_{\substack{i_1 > i_2 > \cdots > i_k \\ i_1, \ldots, i_k \in S}} u_{i_1} u_{i_2} \cdots u_{i_k} \,. \tag{1.2}$$

Again, $e_0(\mathbf{u}_S) = 1$, and $e_k(\mathbf{u}_S) = 0$ unless $0 \le k \le |S|$. (Here $|S|$ is the cardinality of S.)

Theorem 1.1 *(The Rule of Three for noncommutative elementary symmetric functions). Let $\mathbf{u} = (u_1, \ldots, u_N)$ and $\mathbf{v} = (v_1, \ldots, v_N)$ be ordered N-tuples of elements in a ring R.*

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