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Integrable representations of root-graded Lie algebras

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ABSTRACT

In this paper we study the category of representations of a root-graded Lie algebra L which are integrable as representations of a finite-dimensional semisimple subalgebra $\mathfrak{g}$ and whose weights are bounded by some dominant weight of $\mathfrak{g}$. We link this category to the module category of an associative algebra, whose structure we determine for map algebras and $\mathfrak{sl}_n(A)$.

Our approach unifies recent work of Chari and her collaborators on map algebras, of Fourier and Savage and their collaborators on equivariant map algebras, as well as the classical work of Seligman on isotropic Lie algebras.

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Introduction

This paper unites two strands of research which up to now did not have any interaction. Historically, the first strand is the work of Seligman on representations of finite-dimensional isotropic central-simple Lie algebras L over fields $\mathbb{k}$ of characteristic 0 ([40,41]). While these could be studied in the spirit of Tits' approach to representation theory [42] using Galois descent, Seligman pursued "rational methods", i.e., describing the representations over $\mathbb{k}$ (rather than the algebraic closure of $\mathbb{k}$). The Lie algebra L decomposes with respect to a maximal split toral subalgebra $\mathfrak{h} \subset L$ as $L = \bigoplus_{\alpha \in \Theta \cup \{0\}} L_\alpha$ where $\Theta \subset \mathfrak{h}^*$ is an irreducible, possibly non-reduced root system and the L_α are the root spaces of L with respect to $\mathfrak{h}$. Any finite-dimensional irreducible representation of L has weights bounded by a dominant integral weight λ of Θ. Seligman introduced a unital associative algebra $\mathbb{S}^\lambda$ (see Definition 4.2) and linked the finite-dimensional irreducible representations of L and of $\mathbb{S}^\lambda$.

The second strand is essentially due to Chari and her collaborators. It concerns integrable representations of map algebras, i.e., Lie algebras $L = \mathfrak{g} \otimes_{\mathbb{C}} A$ where $\mathfrak{g}$ is a finite-dimensional simple complex Lie algebra and A is a unital commutative associative $\mathbb{C}$-algebra. (The name "map algebra" comes from the interpretation of L as regular maps from the affine scheme $\operatorname{Spec}(A)$ to the affine variety $\mathfrak{g}$.) While originally $A = \mathbb{C}[t^{\pm 1}]$ (loop algebras [16]) or $A = \mathbb{C}[t]$ (current algebras [15]), the algebra A was soon taken to be arbitrary, see for example [18] for an earlier paper. Note that the Lie algebra $L = \mathfrak{g} \otimes_{\mathbb{C}} A$ decomposes in a similar way as Seligman's Lie algebras: Let Δ be the root system of $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ so that $\mathfrak{g} = \bigoplus_{\alpha \in \Delta \cup \{0\}} \mathfrak{g}_\alpha$. Then $L = \bigoplus_{\alpha \in \Delta} L_\alpha$ where $L_\alpha = \mathfrak{g}_\alpha \otimes_{\mathbb{C}} A$ are the weight spaces of L under the canonical $\mathfrak{h}$-action. Regarding the representation theory of map algebras, a big step forward was undertaken in the paper [13] by Chari–Fourier–Khandai, which stressed a categorical point of view. Denote by $\mathcal{I}^\lambda(L, \mathfrak{g})$ the category of integrable representations of $L = \mathfrak{g} \otimes_{\mathbb{C}} A$ whose weights with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ are bounded by a dominant weight λ for $(\mathfrak{g}, \mathfrak{h})$. It was shown in [13], among many other things, that $\mathcal{I}^\lambda(L, \mathfrak{g})$ is closely related to the module category of a commutative associative $\mathbb{C}$-algebra $\mathbf{A}_\lambda$ (see Definition 3.10). Moreover, the algebra $\mathbf{A}_\lambda$ was determined in the Noetherian case. The paper also introduced global Weyl modules as the initial cyclic objects in $\mathcal{I}^\lambda(L, \mathfrak{g})$, generalizing the case $A = \mathbb{C}[t^{\pm 1}]$ of [16]. It has been the blueprint for many sequels, in which the results of [13] have been generalized to the setting of various equivariant map algebras, culminating with the recent paper [20].

The starting point of this paper is the observation that map algebras $\mathfrak{g} \otimes A$ and the equivariant map algebras studied in the sequels to [13] like for example [20], as well

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