Accepted Manuscript

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PII: S0021-8693(17)30198-9
DOI: http://dx.doi.org/10.1016/j.jalgebra.2017.03.016
Reference: YJABR 16161

To appear in: *Journal of Algebra*

Received date: 30 September 2016

Please cite this article in press as: N. Gordeev et al., Word maps, word maps with constants and representation varieties of one-relator groups, *J. Algebra* (2017), http://dx.doi.org/10.1016/j.jalgebra.2017.03.016

WORD MAPS, WORD MAPS WITH CONSTANTS AND REPRESENTATION VARIETIES OF ONE-RELATOR GROUPS

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To Efim Zelmanov on the occasion of his 60th birthday

Abstract. We consider word maps and word maps with constants on a simple algebraic group G. We present results on the images of such maps, in particular, we prove a theorem on the dominance of "general" word maps with constants, which can be viewed as an analogue of a well-known theorem of Borel on the dominance of genuine word maps. Besides, we establish a relationship between the existence of unipotents in the image of the map induced by $w \in F_m$ and the structure of the representation variety $R(\Gamma_w, G)$ of the group $\Gamma_w = F_m/\langle w\rangle$.

Introduction

Word maps. Let F_m be the free group of rank m. Fix its generators $x_1, \ldots, x_m$. Then for any word $w = w(x_1, \ldots, x_m) \in F_m$ and any group G one can define the *word map*

$$\widetilde{w}\colon G^m \to G$$

by evaluation. Namely, $\widetilde{w}(g_1, \ldots, g_m)$ is obtained by substituting g_i in place of x_i and g_i^{-1} in place of x_i^{-1} followed by computing the resulting value $w(g_1, \ldots, g_m)$.

Word maps have been intensely studied over at least two past decades in various contexts (see, e.g., [Se], [Shal], [BGK], [KBKP] for surveys). In this paper, we consider the case where $G = \mathcal{G}(K)$ is the group of K-points of a simple linear algebraic group $\mathcal{G}$ defined over an algebraically closed field K. We are mainly interested in studying the image of $\widetilde{w}$. Borel's theorem [Bo1] says that $\widetilde{w}$ is dominant, i.e., its image contains a Zariski dense open subset of G. However, $\widetilde{w}$ may not be surjective: this may happen in the case of power maps on groups with non-trivial centre (say, squaring map on $\mathrm{SL}(2, \mathbb{C})$) and, if $\mathcal{G}$ is not of type A, even on adjoint groups, see [Ch1], [Ch2], [Stei]. For the adjoint groups of type A, the surjectivity problem is wide open, even in the case of groups of rank 1, and even for words in two variables.

The goal of the present paper is two-fold. First, we extend our viewpoint on the dominance and surjectivity problems from genuine word maps to word maps with constants and establish a partial, "generic" analogue of Borel's dominance theorem. Another extension concerns a continuation of the word map $\widetilde{w}\colon \mathrm{GL}_n(K)^m \to \mathrm{GL}_n(K)$ to the map $\widetilde{w}^*\colon \mathrm{M}_n(K)^m \to \mathrm{M}_n(K)$. Being interesting in its own right, this method yields, as a by-product, a new proof of some results of Bandman and Zarhin [BZ], who proved the surjectivity of $\widetilde{w}$ for $G = \mathrm{PGL}_2(K)$ in the case where K is an algebraically closed field of characteristic zero, $m = 2$, and $w \in F_m \setminus F_m^2$, where $F_m^1 = [F_m, F_m], \ldots, F_m^i = [F_m^{i-1}, F_m^{i-1}], \ldots$.

Our second goal consists in studying the geometric structure of the representation variety of the one-relator group $\Gamma_w := F_m/\langle w\rangle$ with an eye towards applying the data on its