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# Character varieties as a tensor product

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ABSTRACT

In this short note we show that representation and character varieties of discrete groups can be viewed as tensor products of suitable functors over the PROP of cocommutative Hopf algebras. Such view point has several interesting applications. First, it gives a straightforward way of deriving the functor sending a discrete group to the functions on its representation variety, which leads to representation homology. Second, using a suitable deformation of the functors involved in this construction, one can obtain deformations of the representation and character varieties for the fundamental groups of 3-manifolds, and could lead to better understanding of quantum representations of mapping class groups.



## 1. Introduction

Generally speaking, representation variety (more precisely, scheme) is the scheme parametrizing homomorphisms of one algebraic object into another one. For example, there are representation schemes $\mathsf{Rep}(A, V)$ parametrizing homomorphisms of associative algebras $A \to \mathrm{End}(V)$; or the scheme $\mathsf{Rep}(\Gamma, G)$ parametrizing group homomorphisms from a discrete group $\Gamma$ to an algebraic group $G$; or the scheme $\mathsf{Rep}(\mathfrak{a}, \mathfrak{g})$ of Lie algebra

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homomorphisms $\mathfrak{a} \to \mathfrak{g}$. In each of the examples, there is a natural group acting on these schemes, and the character variety is the (categorical) quotient of the representation variety by the action of the group. For example, $\mathrm{GL}(V)$ acts on $\mathsf{Rep}(A, V)$ by conjugation, $G$ acts on $\mathsf{Rep}(\Gamma, G)$, and $\mathsf{Rep}(\mathfrak{a}, \mathfrak{g})$ has the natural action of an algebraic group $G$ with $\mathfrak{g} = \mathcal{L}ie(G)$. The geometry of representation and character varieties has been heavily studied (see, for example, [25],[32] and references therein). Character varieties play an important role in knot theory ([16],[31],[12] to name a few references), moduli of connections (see, for example, [7],[19]), they give interesting invariants called representation homology (see [9], and Section 5).

In this paper we are trying to advocate a slightly different point of view on constructing representation and character varieties. Let $\mathcal{H}$ be the PROP of cocommutative Hopf algebras, that is, a small category with $\mathrm{Ob}(\mathcal{H}) = \mathbb{N}$ and with $\mathrm{Mor}(\mathcal{H})$ generated by certain morphisms, suggestively denoted by $m, \Delta, \eta, \varepsilon, S, \tau$. For a precise definition, see Section 2. There is a natural correspondence between strict monoidal functors $\mathcal{H} \to \mathcal{V}ect$ and cocommutative Hopf algebras. For any discrete group $\Gamma$, there is a natural cocommutative Hopf algebra associated to it, namely, its group algebra $k[\Gamma]$. The group algebra, in turn, determines a functor $k[\Gamma]\colon \mathcal{H} \to \mathcal{V}ect$ sending $[n]$ to $k[\Gamma]^{\otimes n}$. Similarly, the algebra of regular functions $k(G)$ on an affine algebraic group $G$ is a *commutative* Hopf algebra, and therefore determines a functor $k(G)\colon \mathcal{H}^{op} \to \mathcal{V}ect$. The following is one of the main results of this note.

**Theorem 4.1.** *There are natural isomorphisms of commutative algebras*

1. $[k(G)] \otimes_{\mathcal{H}} k[\Gamma] \simeq k(\mathsf{Rep}(\Gamma, G))$;
2. *if* $\Gamma_1, \Gamma_2$ *are two discrete groups, then* $\mathrm{Hom}^{\mathcal{H}}(k[\Gamma_1], k[\Gamma_2]) \simeq k[\mathsf{Rep}(\Gamma_1, \Gamma_2)]$, *where* $k[\mathsf{Rep}(\Gamma_1, \Gamma_2)]$ *denotes the algebra of functions with finite support on the discrete set* $\mathsf{Rep}(\Gamma_1, \Gamma_2)$.

In the theorem, $[k(G)] \otimes_{\mathcal{H}} k[\Gamma]$ denotes the tensor product of two functors, defined by the formula (1). By $\mathrm{Hom}^{\mathcal{H}}$ we denote the space of all ($k$-linear) natural transformations between two functors.

It is important to note that this theorem is not really useful for computations. Indeed, unraveling the definition of the tensor product over $\mathcal{H}$ one is quickly lead to a quotient of the space of functions on several copies of $G$ by the relations coming from a presentation of the group $\Gamma$.

The result essentially equivalent to the first part of this Theorem was independently obtained by Massuyeau–Turaev, see [29]. They used that both functors $[k(G)]$ and $k[\Gamma]$ are monoidal, which turns the tensor product over $\mathcal{H}$ into a quotient of the tensor algebra of $k(G) \otimes k[\Gamma]$. One of the advantages of our viewpoint is that it can be applied even to non-monoidal functors. In particular, this construction also allows to view the *character* variety as the tensor product over $\mathcal{H}$ of the functor $k[\Gamma]$ and the functor $[k(G)]^G\colon \mathcal{H}^{op} \to \mathcal{V}ect$ sending $[n]$ to $[k(G^n)]^G$, the algebra of invariants under the diagonal action of $G$