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Possible indices for a class of association schemes with thin residue equal to the thin radical

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ABSTRACT

We consider association schemes with thin radical isomorphic to an elementary abelian p-group of rank 2, such that the thin residue coincides with the thin radical and all non-thin elements have valency p. We show that when the order of the thin quotient of such schemes exceeds p^2, it must be equal to $p^2 + p + 1$ (a known upper bound), and we show that there exist examples whose thin quotient has order $p^2 - 1$. We also show that if the thin quotient has order at least 10, then one can always find a non-Schurian example with the same thin quotient.

1. Introduction

Association schemes are combinatorially defined objects that can be viewed as generalizations of groups. Indeed, all groups may be viewed as “thin” association schemes, but association schemes have certain desirable properties which groups lack; e.g. while the quotient of a group by a subgroup need not be a group, one can take arbitrary quotients of an association schemes by a closed subset (a kind of subobject).

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One often studies an association scheme by considering the largest closed subset which comes from a group (the thin radical), or the smallest closed subset such that corresponding quotient is a group (the thin residue). The quotient of a scheme by its thin residue is then called its thin quotient.

It is especially interesting when the thin residue coincides with the thin radical; such schemes are in a sense the first outpost of association schemes outside of group theory. In particular, if one begins with a class of groups, it is natural to consider association schemes having a group in this class as both thin residue and thin radical; one might also ask that the relations outside the thin radical be as close to thin as possible.

In [4], Hirasaka and Kim consider schemes S with thin residue equal to the thin radical, with this radical corresponding to the elementary abelian p-group of rank 2. For such schemes, the valencies of all elements must be a power of p; thus, in order for the non-thin elements to be as close to thin as possible, they require all non-thin elements to have valency p. Let $\mathcal{S}^p$ denote the class of such schemes. They call the order of the thin quotient the *index* of the thin residue in S.

In [1], Cho, Hirasaka and Kim had already proved that any scheme in $\mathcal{S}^p$ has index $\delta \leq p^2 + p + 1$. In [4], schemes in $\mathcal{S}^p$ with index p^2 and $p^2 + p + 1$ are constructed, as well as schemes of index between 3 and $p + 2$ inclusive.

The authors also consider the question of Schurity. A scheme is Schurian if the automorphisms of the scheme act transitively on each of the relations of the scheme. Such schemes are of interest because they are precisely the schemes obtained as quotients of thin schemes, and also because they can be obtained as the set of orbitals of a transitive group action. Zieschang [8] proved that association schemes whose thin residue is thin and corresponds to a cyclic group are all Schurian, so the next natural case to consider is the schemes in $\mathcal{S}^p$. Hirasaka and Kim proved that for $4 \leq \delta \leq p + 2$, there exist non-Schurian schemes in $\mathcal{S}^p$ with index δ, and that the schemes of index p^2 and $p^2 + p + 1$ which they constructed are Schurian.

In this paper, we show that if $p^2 < \delta < p^2 + p + 1$, then there do not exist any schemes in $\mathcal{S}^p$ with index δ, and we show that the thin quotient of any scheme in $\mathcal{S}^p$ with index p^2 must be elementary abelian of rank 2. We show that there is a scheme in $\mathcal{S}^p$ with index $p^2 - 1$. We prove that if a scheme in $\mathcal{S}^p$ has odd index δ, with $p^2 - p + 2 \leq \delta < p^2$, then $\delta - (p^2 - p)$ must be a divisor of $p - 1$. Finally, we prove that if there is a scheme S in $\mathcal{S}^p$ whose index is at least 10, then there exists a non-Schurian scheme S' in $\mathcal{S}^p$ whose thin quotient is isomorphic to that of S.

Our paper is organized as follows: in Section 2, we give background and definitions about association schemes. More details about the theory of association schemes can be found in [6].

In Section 3, we consider functions from the set of nonidentity elements in a group G to a fixed finite set X, satisfying a certain condition (see Definition 3.2); we call these X*-valued transition functions* on G. The most important case is when $X = \mathcal{U}_p$, the set of nontrivial cyclic subgroups of $C_p \times C_p$. We give several examples of such transition functions in Subsection 3.1; in particular, we give an example of a $\mathcal{U}_p$-valued