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Control of fixed points over discrete p-toral groups, and existence and uniqueness of linking systems

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The existence and uniqueness of linking systems associated to saturated fusion systems over discrete p-toral groups were proved by Levi and Libman. Their proof makes indirect use of the classification of the finite simple groups. Here we extend some results and arguments of Glauberman and Lynd to show how to remove this assumption.

The notion of p-local compact groups is due to Broto, Levi and Oliver [3]. It encodes in an algebraic way the p-local homotopy theory of compact Lie groups, p-compact groups, and some other families of similar nature. In particular, the theory of p-local compact groups includes the theory of p-local finite groups (see [2] or [1, Chapter III]). Specifically, a p-local compact group is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a discrete p-toral group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is an associated centric linking system. One basic question about p-local compact groups is the following: given a saturated fusion system

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over a p-toral group S, is there an associated linking system, and if so, is this linking system unique?

In the theory of p-local finite groups, Chermak was the first to succeed in answering these questions. In [4], using the theory of partial groups and localities, he shows that, given a saturated fusion system over a finite p-group S, there exists an associated centric linking system which is unique up to isomorphism. His theory gives also a totally new and more group-theoretic approach to p-local finite groups. Later, Oliver [9] translated Chermak's proof into the language of obstruction theory, and then Levi and Libman [7] extended Oliver's proof to p-local compact groups. Thus, we have the following theorem.

Theorem 1 *([7, Theorem B]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group. Then there exists a centric linking system $\mathcal{L}$ associated to $\mathcal{F}$, and $\mathcal{L}$ is unique up to isomorphism.*

Chermak's proof of the existence and uniqueness of centric linking systems uses indirectly the classification of finite simple groups. In the same way, the proof of Oliver and the extension to p-local compact groups by Levi and Libman also rely on the classification of finite simple groups. Recently, Glauberman and Lynd [5] succeeded, using a result of Glauberman [6] on control of fixed points, in giving a classification-free proof of the p-local finite version of Theorem 1.

In this article, we present an extension of arguments of Glauberman and Lynd to the more general setup of p-local compact groups, and thus completing the process of removing the classification of finite simple groups from the proof of Theorem 1.

Oliver's obstruction theory, and its extension to p-local compact groups by Levi and Libman, give a link between the existence and uniqueness of linking systems and a vanishing property of higher limits of a certain functor. For $\mathcal{F}$ a saturated fusion system over a discrete p-toral group S, let $\mathcal{O}(\mathcal{F}^c)$ be the associated centric orbit category and let $\mathcal{Z} : \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$ denote the contravariant functor that associates to a subgroup its center. The existence and uniqueness of centric linking systems can be translated as follows in terms of obstruction theory.

Proposition 1 *([7, Proposition 1.7]). Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. An associated centric linking system exists if $\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^3 \mathcal{Z} = 0$, and if also $\varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^2 \mathcal{Z} = 0$, then that linking system is unique up to an isomorphism.*

Using the ideas of Chermak, Oliver [9] in the p-local finite group case, and Levi and Libman [7] in the general case, proved the following theorem which implies Theorem 1.

Theorem 2. *Let $\mathcal{F}$ be a saturated fusion system over a discrete p-toral group S. Then, for all $i > 1$ if $p = 2$, and all $i > 0$ otherwise,*