Journal of Algebra 498 (2018) 38–46

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Groups of automorphisms with TNI-centralizers ☆

Gülin Ercan [a,*], İsmail Ş. Güloğlu [b]

[a] Department of Mathematics, Middle East Technical University, Ankara, Turkey
[b] Department of Mathematics, Doğuş University, Istanbul, Turkey

ARTICLE INFO

*Article history:*
Received 23 February 2017
Available online 20 November 2017
Communicated by E.I. Khukhro

*MSC:*
20D10
20D15
20D45

*Keywords:*
TNI-subgroup
Automorphism
Centralizer
Frobenius group

ABSTRACT

A subgroup $H$ of a finite group $G$ is called a $TNI$-subgroup if $N_G(H) \cap H^g = 1$ for any $g \in G \setminus N_G(H)$. Let $A$ be a group acting on $G$ by automorphisms where $C_G(A)$ is a $TNI$-subgroup of $G$. We prove that $G$ is solvable if and only if $C_G(A)$ is solvable, and determine some bounds for the nilpotent length of $G$ in terms of the nilpotent length of $C_G(A)$ under some additional assumptions. We also study the action of a Frobenius group $FH$ of automorphisms on a group $G$ if the set of fixed points $C_G(F)$ of the kernel $F$ forms a $TNI$-subgroup, and obtain a bound for the nilpotent length of $G$ in terms of the nilpotent lengths of $C_G(F)$ and $C_G(H)$.


## 1. Introduction

We call a subgroup $H$ of a group $G$ a *trivial normalizer intersection subgroup* (simply a *TNI-subgroup*) *of* $G$ if $N_G(H) \cap H^g = 1$ for any $g \in G \setminus N_G(H)$. Clearly, a normal subgroup is a $TNI$-subgroup. This concept was introduced in the work [2] of Hering in which he studied a finite group $G$ containing a $TNI$-subgroup $H$ and determined the structure of $H^G$ in case $|H|$ is even and $H^G$ is nonsolvable.

☆ This work has been supported by the Research Project TÜBİTAK 114F223.
* Corresponding author.
*E-mail addresses:* ercan@metu.edu.tr (G. Ercan), iguloglu@dogus.edu.tr (İ.Ş. Güloğlu).

https://doi.org/10.1016/j.jalgebra.2017.10.021
0021-8693/

In the present paper we investigate the structure of a finite group $G$ admitting a group $A$ as a group of automorphisms such that the subgroup $C_G(A)$ of fixed points of $A$ in $G$ is a $TNI$-subgroup of $G$. This can be regarded as a generalization of the so-called fixed point free action which assumes that $C_G(A) = 1$. In this case, if $(|G|, |A|) = 1$ or $A$ is nilpotent, it is known that the group $G$ is solvable. We denote the nilpotent length of the solvable group $G$ by $f(G)$. In the coprime case if for every proper subgroup $A_0$ of $A$ and for every irreducible elementary abelian $A_0$-invariant section $V$ of $G$, there exists $v \in V$ such that $C_{A_0}(v) = C_{A_0}(V)$, then the nilpotent length of $G$ is bounded by the length $\ell(A)$ of the longest chain of subgroups of $A$ (see [7] for a survey until then). There is a longstanding conjecture in this research area:

*Let $A$ and $G$ be finite groups such that $A$ acts fixed point freely on $G$ by automorphisms. Assume that either the action is coprime or $A$ is nilpotent. Then $f(G) \leq \ell(A)$.*

This problem is still open even in the coprime case. Although it seems rather out of place to look at the much more general situation where $C_G(A)$ is a TNI-subgroup, Hering's work encouraged us to investigate the structure of such a group $G$. As a first result along these lines in Section 3 we prove

**Theorem A.** *Let $A$ be a finite group that acts coprimely on the finite group $G$ by automorphisms. If $C_G(A)$ is a solvable $TNI$-subgroup of $G$, then $G$ is solvable.*

By [6], in case where $A$ is of prime order $f(G)$ may exceed $f(C_G(A))$ at most by two. In the remainder of this section under the condition that $C_G(A)$ is a $TNI$-subgroup we obtain

**Theorem B.** *Let $A$ be a coprime automorphism of prime order of a finite solvable group $G$ such that $C_G(A)$ is a $TNI$-subgroup of $G$. Then $f(G) \leq f(C_G(A)) + 1$. In particular, $f(G) \leq 4$ when $C_G(A)$ is nonnormal.*

This can be regarded as a generalization of the well known result due to Thompson which asserts that a finite group admitting a fixed point free automorphism of prime order is nilpotent.

As an immediate consequence of Theorem B we observe the following bound for the nilpotent length of a solvable group $G$ acted on by a group $A$ coprimely in such a way that $C_G(B)$ is a $TNI$-subgroup of $G$ for any nontrivial proper subgroup $B$ of $A$.

*Let $A$ be a solvable group acting coprimely on the finite solvable group $G$ by automorphisms in such a way that $C_G(B)$ is a $TNI$-subgroup of $G$ for any subgroup $B$ of $A$. Then $f(G) \leq f(C_G(A)) + \ell(A)$. In particular $f(G) \leq \ell(A) + 3$ whenever $C_G(A)$ is nonnormal. (Corollary 3.1)*

Notice that under this hypothesis if one assumes additionally that $C_G(A) = 1$ then $f(G) \leq \ell(A)$ as claimed in the above conjecture.