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Geometry of the word problem for 3-manifold groups

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We provide an algorithm to solve the word problem in all fundamental groups of 3-manifolds that are either closed, or compact with (finitely many) boundary components consisting of incompressible tori, by showing that these groups are autostackable. In particular, this gives a common framework to solve the word problem in these 3-manifold groups using finite state automata.

We also introduce the notion of a group which is autostackable respecting a subgroup, and show that a fundamental group of a graph of groups whose vertex groups are autostackable respecting any edge group is autostackable. A group that is strongly coset automatic over an autostackable subgroup, using a prefix-closed transversal, is also shown to be autostackable respecting that subgroup. Building on work by Antolin and Ciobanu, we show that a finitely generated group that is hyperbolic relative to a collection of abelian subgroups is also strongly coset automatic relative to each subgroup in the collection. Finally, we show that fundamental groups of compact geometric 3-manifolds, with boundary consisting

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of (finitely many) incompressible torus components, are autostackable respecting any choice of peripheral subgroup.

1. Introduction

One fundamental goal in geometric group theory since its inception has been to find algorithmic and topological characteristics of the Cayley graph satisfied by all closed 3-manifold fundamental groups, to facilitate computations. This was an original motivation for the definition of automatic groups by Epstein, Cannon, Holt, Levy, Paterson, and Thurston [12], and its recent extension to Cayley automatic groups by Kharlampovich, Khoussainov, and Miasnikov [21]. These constructions, as well as finite convergent rewriting systems, provide a solution to the word problem using finite state automata. However, automaticity fails for 3-manifold groups in two of the eight geometries, and Cayley automaticity and finite convergent rewriting systems are unknown for many 3-manifold groups. Autostackable groups, first introduced by the first two authors and Holt in [8], are a natural extension of both automatic groups and groups with finite convergent rewriting systems. In common with these two motivating properties, autostackability also gives a solution to the word problem using finite state automata. In this paper we show that the fundamental group of every compact 3-manifold with incompressible toral boundary, and hence every closed 3-manifold group, is autostackable.

Let G be a group with a finite inverse-closed generating set A. Autostackability is defined using a discrete dynamical system on the Cayley graph $\Gamma := \Gamma_A(G)$ of G over A, as follows. A *flow function* for G with *bound* $K \geq 0$, with respect to a spanning tree T in Γ, is a function Φ mapping the set $\vec{E}$ of directed edges of Γ to the set $\vec{P}$ of directed paths in Γ, such that

(F1): for each $e \in \vec{E}$ the path $\Phi(e)$ has the same initial and terminal vertices as e and length at most K,

(F2): Φ acts as the identity on edges lying in T (ignoring direction), and

(F3): there is no infinite sequence $e_1, e_2, e_3, \ldots$ of edges with each $e_i \in \vec{E}$ not in T and each e_{i+1} in the path $\Phi(e_i)$.

These three conditions are motivated by their consequences for the extension $\widehat{\Phi} : \vec{P} \to \vec{P}$ of Φ to directed paths in Γ defined by $\widehat{\Phi}(e_1 \cdots e_n) := \Phi(e_1) \cdots \Phi(e_n)$, where $\cdot$ denotes concatenation of paths. Upon iteratively applying $\widehat{\Phi}$ to a path p, whenever a subpath of $\widehat{\Phi}^n(p)$ lies in T, then that subpath remains unchanged in any further iteration $\widehat{\Phi}^{n+k}(p)$, since conditions (F1–2) show that $\widehat{\Phi}$ fixes any point that lies in the tree T. Condition (F3) ensures that for any path p there is a natural number n_p such that $\widehat{\Phi}^{n_p}(p)$ is a path in the tree T, and hence $\widehat{\Phi}^{n_p+k}(p) = \Phi^{n_p}(p)$ for all $k \geq 0$. The bound K controls the extent