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The behavior of differential, quadratic and bilinear forms under purely inseparable field extensions

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ABSTRACT

Let F be a field of characteristic p and let E/F be a purely inseparable field extension. We study the group $H_p^{n+1}(F) := \operatorname{coker}(\wp : \Omega_F^n \to \Omega_F^n/\mathrm{d}\Omega_F^{n-1})$ of classes of differential forms under the extension E/F and give a system of generators of $H_p^{n+1}(E/F)$. In the case $p = 2$, we use this to determine the kernel $W_q(E/F)$ of the restriction map $W_q(F) \to W_q(E)$ between the groups of nonsingular quadratic forms over F and over E. We also deduce the corresponding result for the bilinear Witt kernel $W(E'/F)$ of the restriction map $W(F) \to W(E')$, where E'/F denotes a modular purely inseparable field extension.

1. Introduction

One important aspect in the theory of quadratic and bilinear forms is to determine the behavior under field extensions, i.e. we want to determine which nonsingular quadratic forms become hyperbolic and which symmetric bilinear forms become metabolic under a given field extension. The main goal of this work is to classify these forms for

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purely inseparable extensions by using differential forms. For $H_p^{n+1}(F) = \mathrm{coker}(\wp : \Omega_F^n \to \Omega_F^n/\mathrm{d}\Omega_F^{n-1})$ and $\nu_n(F) = \ker(\wp : \Omega_F^n \to \Omega_F^n/\mathrm{d}\Omega_F^{n-1})$, we will determine generating systems for $H_p^{n+1}(E/F)$, $\Omega^n(E'/F)$ and $\nu_n(E'/F)$, where E/F resp. E'/F denotes a purely inseparable extension resp. a modular purely inseparable extension. These results are given in Theorem 3.14 and Theorem 4.1. The kernels will be translated to quadratic and bilinear forms in characteristic 2 by the famous isomorphisms $H_2^{n+1}(F) \cong I^nW_q(F)/I^{n+1}W_q(F)$ and $\nu_n(F) \cong I^n(F)/I^{n+1}(F)$ due to Kato. With this, we will determine the quadratic resp. bilinear Witt kernel for the given extension, which are stated in Theorem 5.3 and Theorem 5.8.

There has already been some progress in determining bilinear and quadratic Witt kernels for some given field extensions in characteristic 2. A nice short overview of some of the known Witt kernels in characteristic 2 can for example be found in [7].

We note that in the case $p = 2$, the same result as our main Theorem 3.14 was computed by Laghribi, Aravire and O'Ryan using somewhat different methods independently of this work.

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2. A short introduction to differential forms

We refer to [1], [6] or [2] (for $p = 2$) for any undefined terminology or any basic facts about differential forms that we do not mention explicitly. We will assume $0 \in \mathbb{N}$.

Let us now start by recalling some basic facts about differential forms. Let F always be a field of characteristic $p > 0$, if not stated otherwise. Further, let Ω_F^1 be the F-vector space of absolute differential 1-forms, i.e. Ω_F^1 is the F-vector space generated by the symbols $\mathrm{d}a$ with $a \in F$, together with the relations $\mathrm{d}(a+b) = \mathrm{d}a + \mathrm{d}b$ and $\mathrm{d}(ab) = a\mathrm{d}b + b\mathrm{d}a$ for all $a, b \in F$. Since $\mathrm{d}(F^p) = 0$ we readily see that the map $\mathrm{d} : F \to \Omega_F^1, a \mapsto \mathrm{d}a$ is a F^p-derivation.

By Ω_F^n we denote the n-exterior power $\bigwedge^n \Omega_F^1$, which means Ω_F^n is a F-vector space generated by the symbols $\mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n$ with $a_1, \ldots, a_n \in F$. With this, the map d can be extended to a F^p-linear map $\mathrm{d} : \Omega_F^{n-1} \to \Omega_F^n, a\mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_{n-1} \mapsto \mathrm{d}a \wedge \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_{n-1}$. The image $\mathrm{d}\Omega_F^{n-1}$ of this map is called the set of exact forms and is additively generated by forms of the type $\mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n$. We further set $\Omega_F^0 := F$ and $\Omega_F^n := 0$ for $n < 0$.

Let us now fix a p-basis $\mathcal{B} = \{b_i \mid i \in I\}$ of F over F^p, i.e. we have $F^p(\mathcal{B}) = F$ and for every finite subset $\{b_{i_1}, \ldots, b_{i_k}\} \subset \mathcal{B}$, we have $[F^p(b_{i_1}, \ldots, b_{i_k}) : F^p] = p^k$, where the second condition is called the p-independence of $\mathcal{B}$. For further details on p-bases see [6]. We may assume that the index set I is well-ordered and transfer this ordering to $\mathcal{B}$ by setting $b_i < b_j$ iff $i < j$. We further define the set $\Sigma_n := \{\sigma : \{1, \ldots, n\} \to I \mid \sigma(i) < \sigma(j) \text{ for } i < j\}$ and equip it with the lexicographic ordering, i.e. we set $\sigma \leq \delta$ iff $\sigma(j) \leq \delta(j)$ and $\sigma(i) = \delta(i)$ for some $j \in \{1, \ldots, n\}$ and all $i < j$. Then it is well known that the logarithmic differential forms $\{\frac{\mathrm{d}b_\sigma}{b_\sigma} \mid \sigma \in \Sigma_n\}$ form a F-basis of Ω_F^n, where we set $\frac{\mathrm{d}b_\sigma}{b_\sigma} := \frac{\mathrm{d}b_{\sigma(1)}}{b_{\sigma(1)}} \wedge \ldots \wedge \frac{\mathrm{d}b_{\sigma(n)}}{b_{\sigma(n)}}$. We further define F_j resp. $F_{<j}$ to be the subfield of F