Journal of Algebra 499 (2018) 229–271

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Some remarks on units in Grothendieck–Witt rings

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ARTICLE INFO

Article history:
Received 12 April 2017
Available online 19 December 2017
Communicated by V. Srinivas

Keywords:
Grothendieck–Witt ring
Multiplicative transfer
Homotopy module
Unit

ABSTRACT

We establish new structures on Grothendieck–Witt rings, including a $GW(k)$-module structure on the unit group $GW(k)^{\times}$ and a presentation of $\underline{GW}^{\times}$ as an infinite $\mathbb{G}_m$-loop sheaf. Even though our constructions are motivated by speculations in stable $\mathbb{A}^1$-homotopy theory, our arguments are purely algebraic.

Contents

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https://doi.org/10.1016/j.jalgebra.2017.12.004
0021-8693/

1. Introduction

The main objects of investigation of this article are the ring-valued functors $X \mapsto GW(X)$ and $X \mapsto \underline{GW}(X)$ and their subfunctors of units $GW^\times(X)$ and $\underline{GW}^\times(X)$. Recall that for a scheme X, $GW(X)$ is the *Grothendieck–Witt* ring of X [11], and for X smooth over a perfect field, $\underline{GW}(X)$ is the *unramified Grothendieck–Witt ring* of X [20, Chapter 3]. The connection is that for X (essentially) smooth local, we have $GW(X) = \underline{GW}(X)$, cf. [21, Theorem A].

Our principal contribution is the following. We show that if k is a field of characteristic not 2, then the group of units $GW^\times(k)$ has a canonical structure of a module over $GW(k)$, related to Rost's multiplicative transfer on $GW(k)$. We use this to give a novel presentation of $GW^\times(k)$, see Proposition 25, and to construct a *homotopy module* T_* such that $T_0 \cong \underline{GW}^\times$. See Appendix A for some recollections regarding homotopy modules.

Organisation We now provide an overview of the article. The remaining subsections of the introduction provide a more leisurely account of some of the key ideas mentioned here.

In Section 2, we recall the results of Rost and his students on multiplicative transfers for the Grothendieck–Witt ring $GW(X)$ [22,12]. Specifically, the multiplicative transfer of Rost is defined using a certain norm functor for modules, also defined by Rost. We show that Rost's norm construction coincides with a more general construction of Ferrand [7], in the situation where both apply.

In Section 3, using this comparison of norm constructions, we show that the assignment $F\acute{e}t/S \ni X \mapsto GW(X)$ defines a Tambara functor. Here $F\acute{e}t/S$ denotes the category of finite étale schemes over S, and by a Tambara functor on this category we mean the evident extension of the notion from [27]; see Definition 8 for details. Using a result of Tambara [27, Theorem 6.1], this also yields an alternative proof that the norm maps extend from $Iso(Bil(\bullet))$ to $GW(\bullet)$.

Section 4 contains our main observation. We show that if k is a field of characteristic not 2, then the group of units $GW^\times(k) \subset GW(k)$ is a module over $GW(k)$, in a unique way that is compatible with the projection formula. By this we mean that if A/k is finite étale, then for $x \in GW(A)$ and $y \in GW^\times(k)$ the following formula holds:

$$y^{tr_{A/k}(x)} = N_{A/k}((y|_A)^x).$$

Note that we write the module structure as "exponentiation". This result is Proposition 22. Uniqueness of the $GW(k)$-module structure follows from the fact that as an abelian group, $GW(k)$ is generated by the traces of finite étale algebras, in fact traces of degree at most 2 extensions suffice. This is explained before Proposition 22. Existence/well-definedness is a consequence of Serre's splitting principle; see Lemma 20. In