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Hochschild homology and cohomology of down–up algebras[☆]

Sergio Chouhy [a], Estanislao Herscovich [b,c,a,*], Andrea Solotar [c,a]

[a] *IMAS, UBA-CONICET, Consejo Nacional de Investigaciones Científicas y Técnicas, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina*
[b] *Institut Fourier, Université Grenoble Alpes, 100 rue des Maths, 38610 Gières, France*
[c] *Departamento de Matemática, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina*

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We present a detailed computation of the cyclic and the Hochschild homology and cohomology of generic and 3-Calabi–Yau homogeneous down–up algebras. This family was defined by Benkart and Roby in [3] in their study of differential posets. Our calculations are completely explicit, by making use of the Koszul bimodule resolution and some arguments similar to those used in [13] to compute the Hochschild cohomology of Yang–Mills algebras.

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* Corresponding author.
E-mail addresses: schouhy@dm.uba.ar (S. Chouhy), Estanislao.Herscovich@univ-grenoble-alpes.fr (E. Herscovich), asolotar@dm.uba.ar (A. Solotar).

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1. Introduction

Motivated by the study of the algebra generated by the up and down operators in the theory of *differential* posets defined independently by R. Stanley in [19] and by S. Fomin in [10], or of *uniform* posets defined by P. Terwilliger in [21], G. Benkart and T. Roby introduced in [3] the notion of *down–up algebras*. Moreover, as noted by G. Benkart in [2], down–up algebras are isomorphic to some particular cases of the family of algebras considered by E. Witten in [25], Section 3, in his study of the state spaces of Chern–Simons gauge theory over SU(2). This is relevant in statistical mechanics, since, according to previous work [24] of the same author, the evaluation of the expectation values of Wilson lines, which is done in terms of the representation theory of the algebras introduced by Witten and the theory of Jones polynomials, can be reduced to the evaluation of a two-dimensional statistical mechanical partition function.

Down–up algebras have been intensively studied in [2], [4], [7], [16], [26] among many other articles, and different kinds of generalizations have been defined. See [6] and [8]. Since the homological invariants of an algebra provide useful tools for its description as well as for the study of its representations, many of their homological properties were studied and in particular, a quite convenient projective resolution of the regular bimodule of a down–up algebra was constructed in general by S. Chouhy and A. Solotar in [9], whereas the case $\gamma = 0$ is a particular case of the bimodule Koszul resolution given by R. Berger and N. Marconnet in [5].

Let $\mathbb{K}$ be a fixed field of characteristic 0. Given parameters $(\alpha, \beta, \gamma) \in \mathbb{K}^3$, the associated *down–up algebra* $A(\alpha, \beta, \gamma)$ is defined as the quotient of the free associative algebra $\mathbb{K}\langle u, d\rangle$ by the ideal generated by the relations

$$\begin{aligned} &d^2u - (\alpha dud + \beta ud^2 + \gamma d), \\ &du^2 - (\alpha udu + \beta u^2d + \gamma u). \end{aligned} \tag{1.1}$$

We shall sometimes denote a particular down–up algebra $A(\alpha, \beta, \gamma)$ just by A to simplify the notation.

Amongst down–up algebras, $A(2, -1, 0)$ is isomorphic to the enveloping algebra of the Heisenberg–Lie algebra of dimension 3, and, for $\gamma \neq 0$, $A(2, -1, \gamma)$ is isomorphic to the enveloping algebra of $\mathfrak{sl}(2, \mathbb{K})$. Moreover, Benkart proved in [2] that any down–up algebra such that $(\alpha, \beta) \neq (0, 0)$ is isomorphic to one of Witten's 7-parameter deformations of $\mathscr{U}(sl(2, \mathbb{K}))$.

Any of these algebras has a PBW basis given by

$$\{u^i(du)^jd^k : i, j, k \in \mathbb{N}_{\geq 0}\}. \tag{1.2}$$

Note that the down–up algebra $A(\alpha, \beta, \gamma)$ can be regarded as a $\mathbb{Z}$-graded algebra where the degrees of u and d are respectively 1 and -1. We shall refer to this grading as *special*,