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Invariant differential operators in positive characteristic

Sofiane Bouarroudj [a,*], Dimitry Leites [a,b]

[a] *New York University Abu Dhabi, Division of Science and Mathematics, P.O. Box 129188, United Arab Emirates*
[b] *Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden*

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A B S T R A C T

In 1928, at the IMC, Veblen posed the problem: classify invariant differential operators between spaces of "natural objects" (in modern terms: either tensor fields, or jets) over a real manifold of any dimension. The problem was solved by Rudakov for unary operators (no nonscalar operators except the exterior differential); by Grozman for binary operators. In dimension one, Grozman discovered an indecomposable selfdual operator of order 3 that does not exist in higher dimensions. We solve Veblen's problem in the 1-dimensional case over any field of positive characteristic. Unary invariant operators are known: these are the exterior differential and analogs of the Berezin integral. We construct new binary operators from these analogs and discovered two more (up to dualizations) types of new indecomposable operators of however high order: analogs of the Grozman operator and a completely new type of operators. Gordan's transvectants, aka Cohen–Rankin brackets, always invariant with respect to the simple 3-dimensional Lie algebra, are also invariant, in characteristic 2, with respect to the whole Lie algebra of vector fields on the line when the height of the indeterminate is equal to 2.

* Corresponding author.
E-mail addresses: sofiane.bouarroudj@nyu.edu (S. Bouarroudj), mleites@math.su.se (D. Leites).

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1. Introduction

1.1. Discussion of open problems

In the representation theory of a given algebra or group, the usual ultimate goal is a clear description of its irreducible modules to begin with, description of indecomposables being the next step. Other goals are sometimes also natural and reachable, cf. those discussed here and in [1]. The history of mathematics shows that problems with a nice (at least in some sense, for example, short) answer often turn out to be "reasonable". I. Gelfand used to say that, the other way round, easy-to-formulate lists are answers to "reasonable" problems and advised to try to formulate such problems once we got a short answer.

Hereafter, $\mathbb{K}$ is any field of characteristic $p > 0$, unless otherwise specified.

Speaking of Lie algebras or superalgebras, and their representations over $\mathbb{K}$, which of them is it natural to consider? In [18], Deligne suggests that one should begin with restricted ones: unlike nonrestricted ones, the restricted Lie (super)algebras correspond to groups, in other words: to geometry. Certain problems concerning nonrestricted algebras are also natural (although tough), and have a short answer, e.g., classification of simple Lie algebras for $p > 3$ (for its long proof, see [26,2]).

Rudakov and Shafarevich [24] were the first to describe ALL, not only restricted, irreducible representations of $\mathfrak{sl}(2)$ for $p > 2$. Dolotkazin solved the same problem for $p = 2$, see [10]; more precisely, he described the irreducible representations of the simple 3-dimensional Lie algebra $\mathfrak{o}^{(1)}(3) = [\mathfrak{o}(3), \mathfrak{o}(3)]$. The difference of Dolotkazin's problem from that considered in [24] is that, unlike $\mathfrak{sl}(2)$ for $p > 2$, the Lie algebra $\mathfrak{o}^{(1)}(3)$ is not restricted. These two results show that the description of all irreducible representations looks feasible, to an extent, at least for Lie (super)algebras with indecomposable Cartan matrix or their simple "relatives" (for the classification of both types of Lie (super)algebras over $\mathbb{K}$, see [7]).

1.1.1. Veblen's problem

In 1928, at the IMC, O. Veblen formulated a problem, later reformulated in more comprehensible terms by A. Kirillov and further reformulated as a purely algebraic problem by J. Bernstein who interpreted Rudakov's solution of Veblen's problem for unary operators; for setting in modern words and review, in particular, for a superization of Veblen's problem, see [15].

Let $\mathfrak{vect}(m)$ be the Lie algebra of polynomial vector fields (over a ground field of characteristic 0, say $\mathbb{C}$). For the definition of the $\mathfrak{vect}(m)$-module $T(V)$ of (formal) tensor fields of type V, where V is a $\mathfrak{gl}(m)$-module with lowest weight vector, see [15]. Let us briefly recall the results concerning the **nonscalar** unary and binary invariant differential operators between spaces of tensor fields, although we only need the simplest version of spaces $T(V)$, namely, the spaces of weighted densities $\mathcal{F}_a := T(a\ \mathrm{tr})$, where tr is the 1-dimensional $\mathfrak{gl}(m)$-module given by the trace (supertrace for $\mathfrak{gl}(m|k)$) and $a \in \mathbb{C}$ (or