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κ-Existentially closed groups ☆

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ABSTRACT

Let κ be an infinite cardinal. The class of κ-existentially closed groups is defined and their basic properties are studied. Moreover, for an uncountable cardinal κ, uniqueness of κ-existentially closed groups are shown, provided that they exist. We also show that for each regular strong limit cardinal κ, there exists κ-existentially closed groups. The structure of centralizers of subgroups of order less than κ in a κ-existentially group G are determined up to isomorphism namely, for any subgroup $F \leq G_\nu$ in G with $|F| < \kappa$, the subgroup $C_G(F)$ is isomorphic to an extension of $Z(F)$ by G.

1. Introduction

To set the scene: Let $X = \{x_i \ ; \ i \in I\}$ be a non-empty set of *variables* that freely generate the free group $F(X)$. For any group G consider the free product $G * F(X)$. The

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elements $g \in G \leq G * F(X)$ will be called *constants* of the free product, the elements, $\omega \in G * F(X)$ not in G *words* or *terms* over G. For a word $\omega \in G * F(X)$ the statement "$\omega = 1$" will be called an *equation* over G in the variables occurring in ω, the statement "$\omega \neq 1$" an *in-equation*.

A non-empty set or system Σ of equations and in-equations over the group G is *consistent* (or solvable) over G if there exists a group $H \supseteq G$ and a homomorphism $\varphi : G * F(X) \to H$ with $\varphi(g) = g$ for $g \in G$ such that the set $\varphi(\Sigma)$ is simultaneously a true statement in H. Such a homomorphism φ will be referred to as a *solution map* (or a realization) of Σ in H.

Side-stepping the problem whether a given system Σ of the equations and in-equations over G is consistent and actually finding such a group $H \supseteq G$ with a solution map φ for Σ, we follow W.R. Scott [14] in defining, for every infinite cardinal κ, the class $\mathcal{E}_\kappa$ of all κ-existentially closed (κ-e.c.)-groups.

Definition. The group G is *κ-existentially closed* if $|G| \geq \kappa$ and if for every, over G consistent system Σ of equations and in-equations involving fewer than κ constants and variables, there is a solution map $\varphi : G * F(X) \to G$ for Σ in G itself.

Observe that for the infinite cardinals κ and κ' the relation $\kappa < \kappa'$ entails $\mathcal{E}_{\kappa'} \subseteq \mathcal{E}_\kappa$. For the limit ordinal $\lambda = \sup\{\kappa\ ;\ \kappa < \lambda\}$ one has $\mathcal{E}_\lambda = \bigcap_{\kappa<\lambda} \mathcal{E}_\kappa$.

Since the books of Lyndon–Schupp [11] and Higman–Scott [6] came out, the notion of existentially closed group became better known. Here we use the stronger version of κ-existentially closed group alluded to already in the founding papers by W.R. Scott [14] and A. Macintyre [12]. For uncountable cardinals κ, we prove some general results and one which corresponds to Steinitz's characterization of algebraically closed fields by the absolute transcendence degree algebraic closure and characteristic [15]. In particular, we prove the uniqueness theorem for such groups provided that they exist. For the existentially closed groups see [8] also the survey paper of Leinen in [10].

2. Preliminaries

Initially, we shall assume that the classes $\mathcal{E}_\kappa$ are non-empty and study the structure of its members. Existence of κ-existentially closed groups for any given infinite cardinality κ is not clear at the moment, shall show existence for regular strong limit cardinals in Section 4.

Lemma 2.1. *For $G \in \mathcal{E}_\kappa$ and the abstract group A with $|A| < \kappa$ there is a subgroup of G isomorphic to A.*

Proof. Rewrite the multiplication table of A as a system Σ of equations and in-equations over G with 1 as the only constant in the variables $X = \{x_a; a \in A\}$: