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Orderable groups with Engel-like conditions ☆

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ABSTRACT

Let x be an element of a group G. For a positive integer n let $E_n(x)$ be the subgroup generated by all commutators $[...[[y,x],x],\dots,x]$ over $y \in G$, where x is repeated n times. There are several recent results showing that certain properties of groups with small subgroups $E_n(x)$ are close to those of Engel groups. The present article deals with orderable groups in which, for some $n \geq 1$, the subgroups $E_n(x)$ are polycyclic. Let $h \geq 0$, $n > 0$ be integers and G an orderable group in which $E_n(x)$ is polycyclic with Hirsch length at most h for every $x \in G$. It is proved that there are (h,n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*.

1. Introduction

A group G is called an Engel group if for every $x, y \in G$ the equation $[y, x, x, \dots, x] = 1$ holds, where x is repeated in the commutator sufficiently many times depending on x and y. Throughout the paper, we use the left-normed simple commutator notation $[a_1, a_2, a_3, \dots, a_r] = [...[[a_1, a_2], a_3], \dots, a_r]$. The long commutators $[y, x, \dots, x]$, where x

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occurs i times, are denoted by $[y,_i x]$. An element $x \in G$ is (left) n-Engel if $[y,_n x] = 1$ for all $y \in G$. A group G is n-Engel if $[y,_n x] = 1$ for all $x, y \in G$. Given $x \in G$, the subgroup $E_n(x)$ is the one generated by all elements of the form $[y,_n x]$ where y ranges over G. Note that $E_n(x)$ is not the same as the more familiar subnormal subgroup $[G,_n x] = [[G,_{n-1} x], x]$. There are several recent results showing that certain properties of groups with small subgroups $E_n(x)$ are close to those of Engel groups (see for instance [3,4,10]). The present article deals with orderable groups. A group G is called orderable if there exists a full order relation $\leq$ on the set G such that $x \leq y$ implies $axb \leq ayb$ for all $a, b, x, y \in G$, i.e. the order on G is compatible with the product of G. Kim and Rhemtulla proved that any orderable n-Engel group is nilpotent ([5], see also [7]). In the present article we consider orderable groups G such that the subgroup $E_n(x)$ is polycyclic for each $x \in G$. Recall that a group is polycyclic if and only if it admits a finite subnormal series all of whose factors are cyclic. The Hirsch length $h(K)$ of a polycyclic group K is the number of infinite factors in the subnormal series.

Our aim here is to prove the following theorem.

Theorem 1.1. *Let $h \geq 0$, $n > 0$ be integers and G an orderable group in which $E_n(x)$ is polycyclic with $h(E_n(x)) \leq h$ for every $x \in G$. There are (h, n)-bounded numbers h^* and c^* such that G has a finitely generated normal nilpotent subgroup N with $h(N) \leq h^*$ and G/N nilpotent of class at most c^*.*

Note that if $h = 0$, the proof shows that $N = 1$ and our result becomes the theorem of Kim and Rhemtulla.

One tool used in the proof of Theorem 1.1 deserves a special mention. A well-known theorem of Malcev states that a soluble group of automorphisms of a polycyclic-by-finite group is polycyclic [8]. We require the following quantitative variation of Malcev's theorem.

Let N be a polycyclic-by-finite group with $h(N) = h$ and let Γ be a soluble group of automorphisms of N. Then $h(\Gamma) < h^2 + 2h$.

To our surprise, in the literature we did not find any mention of the fact that $h(\Gamma)$ should be bounded in terms of h and so it seems that this has so far gone unnoticed. The author is grateful to Dan Segal for suggesting the proof given here (see Proposition 3.1 in Section 3).

2. Preliminaries

We start with general facts about nilpotent groups and Engel elements. If α is an automorphism (or just an element) of a group G, the subgroup generated by the elements of the form $g^{-1}g^{\alpha}$ with $g \in G$ is denoted by $[G, \alpha]$. It is well-known that the subgroup $[G, \alpha]$ is an α-invariant normal subgroup in G. Throughout, we write $[G,_i \alpha]$ for $[[G,_{i-1} \alpha], \alpha]$.