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Matrix coefficient realization theory of noncommutative rational functions

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A B S T R A C T

Noncommutative rational functions, i.e., elements of the universal skew field of fractions of a free algebra, can be defined through evaluations of noncommutative rational expressions on tuples of matrices. This interpretation extends their traditionally important role in the theory of division rings and gives rise to their applications in other areas, from free real algebraic geometry to systems and control theory. If a noncommutative rational function is regular at the origin, it can be described by a linear object, called a *realization*. In this article we present an extension of the realization theory that is applicable to *arbitrary* noncommutative rational functions and is well-adapted for studying matrix evaluations.

Of special interest are the minimal realizations, which compensate the absence of a canonical form for noncommutative rational functions. The non-minimality of a realization is assessed by obstruction modules associated with it; they enable us to devise an efficient method for obtaining minimal realizations. With them we describe the stable extended domain of a noncommutative rational function and define a numerical invariant that measures its complexity. Using these results we determine concrete size bounds for rational identity testing, construct minimal symmetric realizations and prove an

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effective local–global principle for linear dependence of noncommutative rational functions.

1. Introduction

As the universal skew field of fractions of a free algebra, noncommutative rational functions naturally play a prominent role in the theory of division rings and in noncommutative algebra in general [3,13,32,33]. On the other hand, they also arise in other areas, such as free real algebraic geometry [23,34,22], algebraic combinatorics [21,20], systems theory [6,7,2], automata theory [39,18,10] and free analysis [27,28].

One of the main difficulties about working with noncommutative rational functions is that they lack a canonical form. For noncommutative rational functions analytic at 0 this can be resolved by introducing *linear representations* or *realizations*, as they are called in automata theory and control theory, respectively: if $\mathbb{r}$ is a noncommutative rational function in g arguments with coefficients in a field $\mathbb{F}$ and defined at $(0,\dots,0)$, then there exist $n \in \mathbb{N}$, $\mathbf{c} \in \mathbb{F}^{1\times n}$, $\mathbf{b} \in \mathbb{F}^{n\times 1}$, and $A_j \in \mathbb{F}^{n\times n}$ for $1 \le j \le g$, such that

$$\mathbb{r}(z_1,\dots,z_g) = \mathbf{c}\left(I_n - \sum_{j=1}^{g} A_j z_j\right)^{-1}\mathbf{b}.$$

In general, $\mathbb{r}$ admits various realizations; however, those with minimal n are similar up to conjugation (see Definition 3.4) and thus unique in some sense. Throughout the paper we address these results as the *classical* representation or realization theory and refer to [6,10] as the main sources. Applications of such realizations also appear outside control and automata theory, for example in free probability [1,9]. Of course, this approach leaves out noncommutative rational functions that are not defined at any scalar point, e.g. $(z_1z_2 - z_2z_1)^{-1}$.

One way of adapting to the general case is to consider realizations of noncommutative rational functions over an infinite-dimensional division ring as in [14, Section 7.6] or [15]. The latter paper considers generalized series over an infinite-dimensional division ring and corresponding realization theory. However, in the aforementioned setting of systems theory, free (real) algebraic geometry and free analysis, noncommutative rational functions are applied to tuples of matrices or operators on finite-dimensional spaces. Hence the need for a representation of noncommutative rational functions that is adapted to the matrix setting.

The aim of this paper is to develop a comprehensive realization theory that is applicable to arbitrary noncommutative rational functions and is based on matrix-valued evaluations. The basic idea is to expand a noncommutative rational function in a generalized power series about some matrix point in its domain. Analogously to the classical