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On the table of marks of a direct product of finite groups

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We present a method for computing the table of marks of a direct product of finite groups. In contrast to the character table of a direct product of two finite groups, its table of marks is not simply the Kronecker product of the tables of marks of the two groups. Based on a decomposition of the inclusion order on the subgroup lattice of a direct product as a relation product of three smaller partial orders, we describe the table of marks of the direct product essentially as a matrix product of three class incidence matrices. Each of these matrices is in turn described as a sparse block diagonal matrix. As an application, we use a variant of this matrix product to construct a ghost ring and a mark homomorphism for the rational double Burnside algebra of the symmetric group S_3.

1. Introduction

The table of marks of a finite group G was first introduced by William Burnside in his book *Theory of groups of finite order* [5]. This table characterizes the actions of G on

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transitive G-sets, which are in bijection to the conjugacy classes of subgroups of G. Thus the table of marks provides a complete classification of the permutation representations of a finite group G up to equivalence.

The Burnside ring $B(G)$ of G is the Grothendieck ring of the category of finite G-sets. The table of marks of G arises as the matrix of the mark homomorphism from $B(G)$ to the free $\mathbb{Z}$-module $\mathbb{Z}^r$, where r is the number of conjugacy classes of subgroups of G. Like the character table, the table of marks is an important invariant of the group G. By a classical theorem of Dress [6], G is solvable if and only if the prime ideal spectrum of $B(G)$ is connected, i.e., if $B(G)$ has no nontrivial idempotents, a property that can easily be derived from the table of marks of G.

The table of marks of a finite group G can be determined by counting inclusions between conjugacy classes of subgroups of G [13]. For this, the subgroup lattice of G needs to be known. As the cost of complete knowledge of the subgroups of G increases drastically with the order of G (or rather the number of prime factors of that order), this approach is limited to small groups. Alternative methods for the computation of a table of marks have been developed which avoid excessive computations with the subgroup lattice of G. This includes a method for computing the table of marks of G from the tables of marks of its maximal subgroups [13], and a method for computing the table of marks of a cyclic extension of G from the table of marks of G [12].

The purpose of this article is to develop tools for the computation of the table of marks of a direct product of finite groups G_1 and G_2. The obvious idea here is to relate the subgroup lattice of $G_1 \times G_2$ to the subgroup lattice of G_1 and G_2, and to compute the table of marks of $G_1 \times G_2$ using this relationship. Many properties of $G_1 \times G_2$ can be derived from the properties of G_1 and G_2 with little or no effort at all. Conjugacy classes of elements of $G_1 \times G_2$, for example, are simply pairs of conjugacy classes of G_1 and G_2. And the character table of $G_1 \times G_2$ is simply the Kronecker product of the character tables of G_1 and G_2. However the relationship between the table of marks of $G_1 \times G_2$ and the tables of marks of G_1 and G_2 is much more intricate.

A flavour of the complexity to be expected is already given by a classical result known as Goursat's Lemma (Lemma 2.1), according to which the subgroups of a direct product of finite groups G_1 and G_2 correspond to isomorphisms between sections of G_1 and G_2. This article presents the first general and systematic study of the subgroup lattice of a direct product of finite groups beyond Goursat's Lemma. Only very special cases of such subgroup lattices have been considered so far, e.g., by Schmidt [15] and Zacher [16].

In view of Goursat's Lemma, it seems appropriate to first develop some theory for sections in finite groups. Here, a section of a finite group G is a pair (P, K) of subgroups P, K of G such that K is a normal subgroup of P. We study sections by first defining a partial order $\leqslant$ on the set of sections of G as componentwise inclusion of subgroups: $(P', K') \leqslant (P, K)$ if $P' \leqslant P$ and $K' \leqslant K$. Now, if $(P', K') \leqslant (P, K)$, the canonical homomorphism $P'/K' \to P/K$ decomposes as a product of three maps: an epimorphism, an isomorphism and a monomorphism. We show that this induces a decomposition of the partial order $\leqslant$