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On homological smoothness of generalized Weyl algebras over polynomial algebras in two variables

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ABSTRACT

Homological smoothness and twisted Calabi–Yau property of generalized Weyl algebras over polynomial algebras in two variables is studied. A necessary and sufficient condition to be homologically smooth is given. The Nakayama automorphisms of such algebras are also computed in terms of the Jacobian determinants of defining automorphisms.

1. Introduction

Motivated by the study of algebras analogous with the classical Weyl algebra $A_1(\Bbbk)$ over a field $\Bbbk$, Bavula introduced in [2] the notion of (degree one) generalized Weyl algebras over the polynomial algebra $\Bbbk[z]$. Later on, generalized Weyl algebras over any algebra B were defined in [3]. Roughly speaking, a generalized Weyl algebra over B is an extension of B by two formal variables x, y, parameterized by an automorphism σ on B and a central element $\varphi \in B$, denoted by $B(\sigma, \varphi)$. Many people have intensively studied the situation $B = \Bbbk[z]$, and such generalized Weyl algebras are denoted by $W_{(1)}$ in this

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paper. It is illustrated in [2] that the global dimensions of $W_{(1)}$ are equal to 1, 2, or ∞, and the latter occurs if and only if the defining polynomial φ admits a multiple root (see also [3], [18], [35]). Their Hochschild homology and cohomology were computed in [15], [35]. In particular, a remarkable result in [15] is that to assure a duality between its Hochschild homology and cohomology (with coefficients in $W_{(1)}$ itself), $W_{(1)}$ should have finite projective dimension as a bimodule over itself, or equivalently, be homologically smooth. To answer [25, Question 2] (see also [28]), the author explained in [27] when $W_{(1)}$ is homologically smooth; a necessary and sufficient condition was given, under which a duality between its Hochschild homology and cohomology with coefficients in any bimodule was established. Global dimensions of generalized Weyl algebras $W_{(n)}$ over $B = \Bbbk[z_1, \dots, z_n]$ were also examined by Bavula [1].

Among the results mentioned above and others, B is always required to be commutative. This is not incidental. From an algebraic point of view, noncommutative rings are more rigid than commutative ones, namely, a commutative ring possesses “more” automorphisms and central elements than a noncommutative ring. A typical example is: if B is the generic quantum 2-plane $\Bbbk_q[z_1, z_2]$, then σ is necessarily given by $\sigma(z_1) = az_1$, $\sigma(z_2) = bz_2$ for some nonzero scalars a, $b \in \Bbbk$, and the central element φ must be in $\Bbbk$. One concludes that in this case any generalized Weyl algebra over $\Bbbk_q[z_1, z_2]$ is isomorphic to a localization of a quantum 3-space. Hence the noncommutativity of B usually makes the research of generalized Weyl algebras trivial. So generalized Weyl algebras over commutative rings are more interesting. According to papers on this topic, especially the papers by Bavula, two spaces are extremely important in researching generalized Weyl algebras. One is the space $\mathrm{MaxSpec}(B)/\langle\sigma\rangle$ of orbits where $\langle\sigma\rangle$ is the cyclic group generated by σ acting on the space $\mathrm{MaxSpec}(B)$ of maximum spectrum naturally; the other is the σ-stable space $\{\sigma^n(\varphi) \mid n \in \mathbb{Z}\}$. Normally, many properties, such as global dimensions and irreducible representations, depend on both spaces simultaneously. But there is an exception. It is illustrated in [27] that whether a generalized Weyl algebra $W_{(1)}$ is homologically smooth depends only on φ, also, its Nakayama automorphism depends only on σ.

This paper is a sequel to [27], in which we plan to study the homological smoothness of generalized Weyl algebras $W_{(2)}$ over the polynomial algebra $\Bbbk[z_1, z_2]$. Homological smoothness, which is a noncommutative generalization of smoothness for commutative algebras, plays an important role in homological algebra, mathematical physics, etc. An algebra A is called homologically smooth if A admits a bound resolution by finitely generated projective A^{e}-modules (where $A^e := A \otimes_{\Bbbk} A^{\mathrm{op}}$ is the enveloping algebra), or equivalently, A is isomorphic to a perfect complex in the derived category $\mathcal{D}^b(A^{\mathrm{e}}\text{-}\mathsf{Mod})$. Recall that in [27] our strategy was to construct a free $W_{(1)}^e$-module resolution of $W_{(1)}$ and then to compute cohomology by it. We will follow the idea for $W_{(2)}$ in this paper. Since the notion of homotopy double complex introduced in [27] is useful to construct a free resolution of $W_{(2)}$, we review it briefly in this paper. After that, noncommutative differential 1-forms and derivations on $\Bbbk[z_1, z_2]$ are introduced, and a noncommutative version of Jacobian determinant is defined accordingly. All of these appear in §2.