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A note on module structures of source algebras of block ideals of finite groups

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ABSTRACT

Let b be a block ideal of the group algebra of a finite group G over an algebraically closed field k of prime characteristic p with a defect group P. Some direct summands, as $k[P \times P]$-module, of a source algebra of the block ideal b outside of the inertial group of a maximal b-Brauer pair will be given; their multiplicities are congruent to 1 modulo p.

1. Introduction

Throughout this paper we let k be an algebraically closed field of characteristic p.

Let G be a finite group of order divisible by p and b a block ideal of the group algebra kG. Let P be a defect group of b and (P, e_P) a fixed maximal b-Brauer pair. Let i be

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a source idempotent associated with (P, e_P). We shall examine the $k[P \times P]$-module structure of the source algebra $ikGi$ of the block ideal b.

The direct summands isomorphic to $k[Pg]$ for $g \in N_G(P, e_P)$ are well understood. We quote here the following.

Theorem 1.1 *([4, Theorem 44.3]). With the notation above there is a decomposition of $ikGi$ as a $k[P \times P]$-module*

$$ikGi \simeq \left(\bigoplus_{g \in [N_G(P,e_P)/PC_G(P)]} kPg \right) \bigoplus Z,$$

where $[N_G(P, e_P)/PC_G(P)]$ is a complete set of representatives of left cosets $gPC_G(P)$ in $N_G(P, e_P)$ and Z is isomorphic to a direct sum of modules of the form $k[PxP]$ for some $x \in G \setminus N_G(P)$.

In this note we shall investigate direct summands of $ikGi$ outside $N_G(P)$.

We follow Thévenaz [4] or Külshammer [1] for terminology and notation. All modules over group algebras of finite groups considered are finitely generated left modules. Especially for G a finite group the group algebra kG is a left $k[G \times G]$-module by letting $k[G \times G]$ act on kG from left as follows:

$$(g, h)\alpha = g\alpha h^{-1} \quad (g, h) \in G \times G, \alpha \in kG.$$

Notice that the $k[G \times G]$-module kG is a p-permutation module. We also recall here some notion concerning Brauer pairs, before stating our main theorem.

A pair (Q, f) of a p-subgroup Q of G and a block idempotent $f \in kC_G(Q)$ is called a Brauer pair; it is called a b-Brauer pair if the Brauer correspondent of the block ideal $kC_G(Q)f$ to G coincides with b. A Brauer pair (Q, f) is said to be self-centralizing if $Z(Q)$ is a defect group of the block ideal $kC_G(Q)f$; in this case the block ideal $kC_G(Q)f$ has a unique simple module. A Brauer pair (Q, f) is said to be essential if it is self-centralizing and the inertial quotient $N_G(Q, f)/QC_G(Q)$ has a strongly p-embedded proper subgroup; if this is the case, then given a p-subgroup of the inertial quotient, we can take a strongly p-embedded proper subgroup of the inertial quotient containing it. Our main theorem is the following.

Theorem 1.2. *Let $(T, e_T) \subseteq (P, e_P)$ be an essential b-Brauer pair; there exists a subgroup M of the inertial group $N_G(T, e_T)$ such that $M \geqslant N_P(T)C_G(T)$ and $M/TC_G(T)$ is strongly p-embedded in $N_G(T, e_T)/TC_G(T)$.*

Then for each element $x \in N_G(T, e_T) \setminus M$, it follows that $P \cap {}^xP = T$ and that the $k[P \times P]$-module $k[PxP]$ is isomorphic with a direct summand of the source algebra $ikGi$ with multiplicity congruent to 1 modulo p.