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Tropical critical points of the superpotential of a flag variety

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ABSTRACT

In this paper we investigate the idea of a tropical critical point of the superpotential for the full flag variety of type A. Recall that associated to an irreducible representation of $G = SL_n(\mathbb{C})$ are various polytopes whose integral points parameterize a basis for the representation, e.g. the Gelfand–Zetlin polytope. Such polytopes can be constructed via the theory of geometric crystals by tropicalising a certain function, and in fact, the function involved coincides with the superpotential from the Landau–Ginzburg model for G/B coming from mirror symmetry. In mirror symmetry a special role is played by the critical points of the superpotential, and motivated by this, we give a definition of the tropical critical points and use it to find a canonical point in each polytope. We then characterise the highest weights for which this tropical critical point is integral and therefore corresponds to a basis vector of the corresponding representation. Finally we give an interpretation of the tropical critical point by constructing a special vector in the representation using Borel–Weil theory and conjecturing a correspondence between this vector and the tropical critical point.

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1. Introduction

Let G be a simple complex algebraic group, T a choice of maximal torus and $B \supset T$ a choice of Borel subgroup, with opposite Borel subgroup B_-. Given $\lambda \in P^+$, a dominant integral weight of T, let V_λ be the irreducible representation of G with highest weight λ and V_λ^* the dual representation. Consider the case where $\lambda = 2\rho$, the sum of the positive roots of G. There is a special vector in the representation V_λ^* which has geometric origin and is defined as follows. First recall that Borel–Weil theory gives a geometric construction of V_λ^* as $H^0(G/B, \mathcal{L}_\lambda)$, where $\mathcal{L}_\lambda$ is the line bundle

$$G \times^B \mathbb{C}_{-\lambda} = \{(g, x)\}/(g, x) \sim (gb, \lambda(b)x)$$

(see [22]). If $\lambda = 2\rho$ then $\mathcal{L}_{2\rho}$ happens to be the anti-canonical bundle of G/B, so V_λ^* is given by the global sections of the anti-canonical bundle. Now, there exists a special meromorphic volume form ω on G/B, defined uniquely up to sign. This form was first introduced in [19] where it was defined as a natural generalisation of the unique torus-invariant volume form on a torus inside a toric variety. It is the meromorphic differential form on G/B with simple poles exactly along the divisor given by the union of all the Schubert divisors and all the opposite Schubert divisors, see [14, Section 2]. Similar volume forms also appear more recently in work on mirror symmetry and cluster varieties, see [9,3]. Now if we take the inverse of ω, we get a special global section of the anti-canonical bundle of G/B, and thus a distinguished vector in the representation $V_{2\rho}^*$. We would like to give an interpretation of this special vector.

Example 1.1. In the case $G = SL_2(\mathbb{C})$, we have $G/B \simeq \mathbb{P}^1$ via $\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \mapsto x = \frac{g_1}{g_3}$. The Schubert divisors are given by $x = 0$ and $x = \infty$ and the volume form ω is given by $\frac{dx}{x}$ which has a simple pole at $x = 0$ and $x = \infty$. The representation $V_{2\rho}^*$ is given by $\langle x^2 \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle_{\mathbb{C}}$ and ω^{-1} is given by $x\frac{\partial}{\partial x} \in V_{2\rho}^*$.

Restrict now to the case $G = SL_n(\mathbb{C})$. We will interpret this special section ω^{-1} using the mirror dual Landau–Ginzburg model for G/B. A Landau–Ginzburg model for the full flag variety of type A was first introduced by Givental [8], in the form of a regular