Journal of Algebra 497 (2018) 199–218

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Nearly commuting matrices

Zhibek Kadyrsizova

Department of Mathematics, School of Science and Technology, Nazarbayev University, 53 Kabanbay Batyr Ave, Astana, 010000, Kazakhstan

ARTICLE INFO

Article history:
Received 9 June 2017
Available online 20 November 2017
Communicated by V. Srinivas

Keywords:
Frobenius
Singularities
F-purity
Commuting matrices

ABSTRACT

We prove that the algebraic set of pairs of matrices with a diagonal commutator over a field of positive prime characteristic, its irreducible components, and their intersection are F-pure when the size of matrices is equal to 3. Furthermore, we show that this algebraic set is reduced and the intersection of its irreducible components is irreducible in any characteristic for pairs of matrices of any size. In addition, we discuss various conjectures on the singularities of these algebraic sets and the system of parameters on the corresponding coordinate rings.

1. Introduction and preliminaries

In this paper we study algebraic sets of pairs of matrices such that their commutator is either nonzero diagonal or zero. We also consider some other related algebraic sets. First let us define relevant notions.

Let $X = (x_{ij})_{1\le i,j\le n}$ and $Y = (y_{ij})_{1\le i,j\le n}$ be $n\times n$ matrices of indeterminates over a field K. Let $R = K[X, Y]$ be the polynomial ring in $\{x_{ij}, y_{ij}\}_{1\le i,j\le n}$ and let I denote the ideal generated by the off-diagonal entries of the commutator matrix $XY - YX$ and J denote the ideal generated by the entries of $XY - YX$. The ideal I defines the algebraic

E-mail address: zhibek.kadyrsizova@nu.edu.kz.

https://doi.org/10.1016/j.jalgebra.2017.10.019
0021-8693/

set of pairs of matrices with a diagonal commutator and is called *the algebraic set of nearly commuting matrices*. The ideal J defines the algebraic set of pairs of commuting matrices.

Let u_{ij} denote the (i, j)th entry of the matrix $XY - YX$. Then $I = (u_{ij} \mid 1 \leq i \neq j \leq n)$ and $J = (u_{ij} \mid 1 \leq i, j \leq n)$.

Theorem 1 *([3]). The algebraic set of commuting matrices is irreducible, i.e., it is a variety. Equivalently,* $\mathrm{Rad}(J)$ *is prime.*

The following results are due to A. Knutson [8], when the characteristic of the field is 0, and to H. Young [13] in all characteristics.

Theorem 2 *([8,13]). The algebraic set of nearly commuting matrices is a complete intersection, with the variety of commuting matrices as one of its irreducible components. In particular, the set $\{u_{ij} | 1 \leq i \neq j \leq n\}$ is a regular sequence and the dimension of R/I is $n^2 + n$.*

Theorem 3 *([8,13]). When K has characteristic zero, I is a radical ideal.*

A. Knutson in his paper [8] conjectured that $\mathbb{V}(I)$ has only two irreducible components and it was proved in all characteristics by H. Young in his thesis, [13].

Theorem 4 *([13]). If $n \geq 2$, the algebraic set of nearly commuting matrices has two irreducible components, one of which is the variety of commuting matrices and the other is the so-called skew component. That is, I has two minimal primes, one of which is* $\mathrm{Rad}(J)$.

Let $P = \mathrm{Rad}(J)$ and let Q denote the other minimal prime of I, i.e., $\mathrm{Rad}(I) = P \bigcap Q$. The following conjecture was made in 1982 by M. Artin and M. Hochster.

Conjecture 1. *J is reduced, i.e., $J = P$.*

It was answered positively by Mary Thompson in her thesis in the case of 3×3 matrices.

Theorem 5 *([12]). R/J is a Cohen–Macaulay domain when $n = 3$.*

Now let us go back to algebraic sets of nearly commuting matrices and their irreducible components. First, we take a look at what we have when $n = 1, 2$.

When $n = 1$, everything is trivial. More precisely, $I = P = Q = (0) \subset K[x_{11}, y_{11}]$.

When $n = 2$, without loss of generality we may replace X and Y by $X - x_{22}\mathrm{I}_n$ and $Y - y_{22}\mathrm{I}_n$ respectively. Here I_n is the identity matrix of size n. Denote $x'_{11} = x_{11} - x_{22}$, $y'_{11} = y_{11} - y_{22}$. Then the generators of I are 2 by 2 minors