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# Sums-of-squares formulas over algebraically closed fields

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ABSTRACT

In this paper, we consider whether existence of a sums-of-squares formula depends on the base field. We reformulate the question of existence as a question in algebraic geometry. We show that, for large enough $p$, existence of sums-of-squares formulas over algebraically closed fields is independent of the characteristic. We make the bound on $p$ explicit, and we prove that the existence of a sums-of-squares formula of fixed type over an algebraically closed field is theoretically (though not practically) computable.



## 1. Introduction and summary of methods

Let $F$ be a field of characteristic not 2. A *sums-of-squares formula of type* $[r, s, n]$ *over* $F$ is an identity of the form

$$(x_1^2 + \cdots + x_r^2)(y_1^2 + \cdots + y_s^2) = z_1^2 + \cdots + z_n^2$$

where each $z_i$ is a bilinear expression in the $x$'s and $y$'s over $F$.

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The existence of sums-of-squares formulas has been extensively studied since they arose in relation to real normed division algebras, and Hurwitz [11] [12] posed the general question: for what $r, s, n$ does a sums-of-squares formula of type $[r, s, n]$ exist over a field $F$?

Sums-of-squares formulas were used to prove Hurwitz's theorem that the only real normed division algebras are the real numbers, complex numbers, quaternions, and octonions. They continue to be of interest for their relationship to problems in topology and geometry: they provide immersions of projective space into Euclidean space, and they provide Hopf maps. Over arbitrary fields, sums-of-squares formulas provide examples of compositions of quadratic forms.

A detailed treatment of the historical development of the subject and past results can be found in Shapiro's book [14].

Whether existence of a sums-of-squares formula of a fixed type depends on the base field remains an open question. Existence over arbitrary fields is particularly interesting, because formulas over finite fields can be found using computational methods, and these formulas could then yield formulas in the classical characteristic 0 setting.

Most results on the existence of sums-of-squares formulas have been obtained only for fields of characteristic 0. However, for some very special cases of $r, s, n$, Adem [1] [2] and Yuzvinsky [18] have settled the question of existence over arbitrary fields. For fixed $r$ and $s$, topological and geometric considerations have provided lower bounds on the smallest $n$ for which a sums-of-squares formula of type $[r, s, n]$ can exist over $\mathbb{R}$. Recently, Dugger and Isaksen [6] [5] [7] show these lower bounds are valid over any field $F$, improving on the results of Shapiro and Szyjewski [13]. Xie [17] also contributed in this vein.

In this paper, we provide a new formulation for the question of existence of sums-of-squares formulas as a question in algebraic geometry, defining the variety of sums-of-squares formulas. We use this reformulation to prove that, for any type $[r, s, n]$ and large enough $p$, existence over algebraically closed fields of characteristic 0 and $p$ are equivalent. We also prove that existence of sums-of-squares formulas of type $[r, s, n]$ can be decided by performing finitely many computations, checking finitely many coefficients in finitely many finite fields. We obtain explicit bounds on the degrees and characteristics of these fields. Unfortunately, these bounds are too large to ensure that existence is computable in practice.

We begin by reformulating the question of existence of sums-of-squares formulas as a question in algebraic geometry. This is done by observing that a sums-of-squares formula is given by writing each $z_i$ as an $F$-linear combination of the $x_j y_k$, and the coefficients must satisfy certain polynomial equations. Thus, giving a sums-of-squares formula is equivalent to giving a solution to this set of polynomial equations, and a sums-of-squares formula exists over an algebraically closed field if and only if the corresponding variety is non-empty.

Denote by $A^F_{rsn}$ the coordinate ring corresponding to this set of polynomials. Let $X^F_{rsn} = \text{Spec } A^F_{rsn}$ be the corresponding variety, which we call the *variety of sums-of-*