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Local cohomology and base change ☆

Karen E. Smith

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ABSTRACT

Let $X \xrightarrow{f} S$ be a morphism of Noetherian schemes, with S reduced. For any closed subscheme Z of X finite over S, let j denote the open immersion $X \setminus Z \hookrightarrow X$. Then for any coherent sheaf $\mathcal{F}$ on $X \setminus Z$ and any index $r \geq 1$, the sheaf $f_*(R^r j_* \mathcal{F})$ is generically free on S and commutes with base change. We prove this by proving a related statement about local cohomology: Let R be Noetherian algebra over a Noetherian domain A, and let $I \subset R$ be an ideal such that R/I is finitely generated as an A-module. Let M be a finitely generated R-module. Then there exists a non-zero $g \in A$ such that the local cohomology modules $H^r_I(M) \otimes_A A_g$ are free over A_g and for any ring map $A \to L$ factoring through A_g, we have $H^r_I(M) \otimes_A L \cong H^r_{I \otimes_A L}(M \otimes_A L)$ for all r.

1. Introduction

In his work on maps between local Picard groups, Kollár was led to investigate the behavior of certain cohomological functors under base change [12]. The following theorem directly answers a question he had posed:

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E-mail address: kesmith@umich.edu.

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Theorem 1.1. *Let $X \xrightarrow{f} S$ be a morphism of Noetherian schemes, with S reduced. Suppose that $Z \subset X$ is closed subscheme finite over S, and let j denote the open embedding of its complement U. Then for any coherent sheaf $\mathcal{F}$ on U, the sheaves $f_*(R^r j_* \mathcal{F})$ are generically free and commute with base change for all $r \geq 1$.*

Our purpose in this note is to prove this general statement. Kollár himself had proved a special case of this result in a more restricted setting [12, Thm. 78].

We pause to say precisely what is meant by *generically free and commutes with base change*. Suppose $\mathcal{H}$ is a functor which, for every morphism of schemes $X \to S$ and every quasi-coherent sheaf $\mathcal{F}$ on X, produces a quasi-coherent sheaf $\mathcal{H}(\mathcal{F})$ on S. We say $\mathcal{H}(\mathcal{F})$ is *generically free* if there exists a dense open set S^0 of S over which the $\mathcal{O}_S$-module $\mathcal{H}(\mathcal{F})$ is free. If in addition, for every change of base $T \xrightarrow{p} S^0$, the natural map

$$p^* \mathcal{H}(\mathcal{F}) \to \mathcal{H}(p_X^* \mathcal{F})$$

of quasi-coherent sheaves on T is an isomorphism (where p_X is the induced morphism $X \times_S T \to X$), then we say that $\mathcal{H}(\mathcal{F})$ is *generically free and commutes with base change*. See [12, §72].

Remark 1.2. We do not claim the $r = 0$ case of Theorem 1.1; in fact, it is false. For a counterexample, consider the ring homomorphism splitting $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Q} \twoheadrightarrow \mathbb{Z}$. The corresponding morphism of Noetherian schemes

$$Z = \operatorname{Spec}(\mathbb{Z}) \hookrightarrow X = \operatorname{Spec}(\mathbb{Z} \times \mathbb{Q}) \to S = \operatorname{Spec} \mathbb{Z}$$

satisfies the hypothesis of Theorem 1.1. The open set $U = X \setminus Z$ is the component $\operatorname{Spec} \mathbb{Q}$ of X. The coherent sheaf determined by the module $\mathbb{Q}$ on U is not generically free over $\mathbb{Z}$, since there is no open affine subset $\operatorname{Spec} \mathbb{Z}[\frac{1}{n}]$ over which $\mathbb{Q}$ is a free module. [In this case, the map j is affine, so the higher direct image sheaves $R^p j_* \mathcal{F}$ all vanish for $p > 0$.]

On the other hand, if f is a map of finite type, then the $r = 0$ case of Theorem 1.1 can be deduced from Grothendiecks's Lemma on Generic freeness; see Lemma 4.1.

For the commutative algebraists, we record the following version of the main result, which is essentially just the statement in the affine case:

Corollary 1.3. *Let A be a reduced Noetherian ring. Let R be a Noetherian A-algebra with ideal $I \subset R$ such that the induced map $A \to R/I$ is finite. Then for any Noetherian R module M, the local cohomology modules $H^i_I(M)$ are generically free and commute with base change over A for all $i \geq 0$. Explicitly, this means that there exists an element g not in any minimal prime of A such that the modules $H^i_I(M) \otimes_A A_g$ are free over A_g, and that for any algebra L over A_g, the natural map*