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Groups in which each subgroup is commensurable with a normal subgroup

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A group G is a CN-group if for each subgroup H of G there exists a normal subgroup N of G such that the index $|HN : (H \cap N)|$ is finite. The class of CN-groups contains properly the classes of core-finite groups and that of groups in which each subgroup has finite index in a normal subgroup.
In the present paper it is shown that a CN-group whose periodic images are locally finite is finite-by-abelian-by-finite. Such groups are then described into some details by considering automorphisms of abelian groups. Finally, it is shown that if G is a locally graded group with the property that the above index is bounded independently of H, then G is finite-by-abelian-by-finite.

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1. Introduction and main results

In a celebrated paper, B.H. Neumann [10] showed that for a group G the property that each subgroup H has finite index in a normal subgroup of G (i.e., $|H^G : H|$ is finite) is equivalent to the fact that G has finite derived subgroup (G is *finite-by-abelian*).

A class of groups with a dual property was considered in [1]. A group G is said to be a CF-group (*core-finite*) if each subgroup H contains a normal subgroup of G with finite index in H (i.e., $|H : H_G|$ is finite). As Tarski groups are CF, a complete classification of CF-groups seems to be rather difficult. However, in [1] and [12] it has been proved that *a* CF*-group G whose periodic quotients are locally finite is abelian-by-finite and, if G is periodic, there exists an integer n such that $|H : H_G| \leq n$ for all $H \leq G$* (say that *G is* BCF*, boundedly* CF) and that *a locally graded* BCF*-group is abelian-by-finite*. Furthermore, an easy example of a metabelian (and even hypercentral) group which is CF but not BCF is given. It seems to be a still open question whether every locally graded CF-group is abelian-by-finite. Recall that a group is said to be *abelian-by-finite* if it has an abelian subgroup with finite index and that a group is said to be *locally finite* (*locally graded*, respectively) if each non-trivial finitely generated subgroup is finite (has a proper subgroup with finite index, respectively).

With the aim of considering the above properties in a common framework, recall that two subgroups H and K of a group G are said to be *commensurable* if $H \cap K$ has finite index in both H and K. This is an equivalence relation and will be denoted by $\sim$. Clearly, if $H \sim K$, then $(H \cap L) \sim (K \cap L)$ and $HM \sim KM$ for each $L \leq G$ and $M \lhd G$.

Thus, in the present paper we consider the class of CN*-groups*, that is, *groups in which each subgroup is commensurable with a normal subgroup*. Into details, for a subgroup H of a group G define $\delta_G(H)$ to be the minimum index $|HN : (H \cap N)|$ with $N \lhd G$. Then G is a CN-group if and only if $\delta_G(H)$ is finite for all $H \leq G$. Clearly, subgroups and quotients of CN-groups are also CN-groups.

Note that if a subgroup H of a group G is commensurable with a normal subgroup N, then $S := (H \cap N)_N$ has finite index in H. Thus the class of CN-groups is contained in the class of *sbyf-groups*, that is, groups in which each subgroup H contains a subnormal subgroup S of G such that the index $|H : S|$ is finite (i.e., H is *subnormal-by-finite*). It is known that *locally finite sbyf-groups are (locally nilpotent)-by-finite* (see [7]) and *nilpotent-by-Chernikov* (see [3]).

The extension of a finite group by a CN-group is easily seen to be a CN-group, see Proposition 1.1 below. Moreover, from Proposition 9 in [4] it follows that *for an abelian-by-finite group properties* CN *and* CF *are equivalent*. However, for each prime p there is a nilpotent p-group with the property CN which is neither finite-by-abelian nor abelian-by-finite, see Proposition 1.2.

Our main result is the following.

Theorem A. *Let G be a* CN*-group such that every periodic image of G is locally finite. Then G is finite-by-abelian-by-finite.*