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Root multiplicities for Nichols algebras of diagonal type of rank two

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We determine the multiplicities of a class of roots for Nichols algebras of diagonal type of rank two, and identify the corresponding root vectors. Our analysis is based on a precise description of the relations of the Nichols algebra in the corresponding degrees.

1. Introduction

Since the introduction of Nichols algebras in the late 1990-ies, the topic developed to an own-standing research field with many relationships to different (mainly algebraic or combinatorial) fields in mathematics. In particular, Nichols algebras are heavily used for the study of pointed Hopf algebras. Although Nichols algebras can be defined in any

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suitable braided monoidal category, a big part of the theory is dominated by Nichols algebras of diagonal type.

By now, a deep understanding of the structure of finite-dimensional Nichols algebras of diagonal type is available, based on the existence of a PBW basis [4] and the notion of roots [3]. In the general setting, one is constantly tempted to seek for relationships with Kac–Moody and Borcherds Lie (super) algebras. The latter seems to be very strong in the finite case because of the definitions of real roots in the two theories. However, the knowledge about imaginary roots and their multiplicities is little in the case of Kac–Moody algebras, and even poorer for Nichols algebras of diagonal type. For information on recent activities in the theory of Kac–Moody algebras we refer to [2]. With our results we make a small step towards a better understanding of the Nichols algebra theory in this respect.

In this paper, we concentrate on Nichols algebras of diagonal type of rank two. In order to clarify the context, we introduce the notion of root vector candidates and root vectors. We focus on the special roots $m\alpha_1 + 2\alpha_2$, where $m \in \mathbb{N}_0$ and α_1, α_2 is the standard basis of $\mathbb{Z}^2$. (The root multiplicities of $m\alpha_1 + k\alpha_2$ with $m \in \mathbb{N}_0$, $k \in \{0, 1\}$, have been known before.) We identify the family $(P_k)_{k\in\mathbb{N}_0}$ in the free algebra over a two-dimensional braided vector space V of diagonal type, and relate the relations in the Nichols algebra of V of degree $m\alpha_1 + 2\alpha_2$ to this family. We find two of our results particularly interesting. First, in Proposition 4.3 we prove that if a root vector candidate is a root vector, then any lexicographically larger root vector candidate of the same degree is a root vector, too. Second, in Corollary 4.17 we describe precisely when a root vector candidate is a root vector. This result is based on Theorem 4.16 which we recall here. Let $q = q_{11}$, $r = q_{12}q_{21}$, and $s = q_{22}$, where $(q_{ij})_{1\le i,j\le 2}$ is the braiding matrix of V. Let $\pi : T(V) \to \mathcal{B}(V)$ be the canonical map. Let $\mathbb{J} = \mathbb{J}_{q,r,s} \subseteq \mathbb{N}_0$ be as in Definition 4.12. (In order to determine $\mathbb{J}$ one needs only elementary (and simple) calculations with Laurent polynomials in three indeterminates.) Let $u_k = [x_1^k x_2]$ with $k \ge 0$ be the elements of degree $k\alpha_1 + \alpha_2$ in (6). Define $b_k = \prod_{j=0}^{k-1}(1 - q^j r)$ for any $k \ge 0$. Then $u_k = 0$ in $\mathcal{B}(V)$ if and only if $(k)^!_q b_k = 0$ by Remark 3.10(2). For any $n, k \ge 0$ let $U_n \subseteq \mathcal{B}(V)$ be the subspace in degree $n\alpha_1 + 2\alpha_2$ in (13), and let P_k be the element in Definition 4.6 as a linear combination of the monomials $u_i u_{k-i}$ with $0 \le i \le k$. Then the following hold.

Theorem 1.1. *(See Theorem 4.16.) Assume that* $\operatorname{char}(\Bbbk) = 0$*. Let* $m \in \mathbb{N}_0$ *such that* $(m)^!_q b_m \ne 0$*. Then the elements* $(\operatorname{ad} x_1)^{m-j}(P_j)$ *with* $j \in \mathbb{J} \cap [0, m]$ *form a basis of* $\ker(\pi) \cap U_m$*.*

At the end of the paper we illustrate our results on two examples, each of which is related to a quantized enveloping algebra of an affine Kac–Moody algebra of rank two.

The paper is organized as follows. In Section 2, we give some equations for Gaussian binomial coefficients, which will be needed later. In Section 3, we recall some fundamental definitions and results on which our work is based. In Section 4 we formulate and prove