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The colored symmetric and exterior algebras

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A R T I C L E I N F O

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A B S T R A C T

We study colored generalizations of the symmetric algebra and its Koszul dual, the exterior algebra. The symmetric group $\mathfrak{S}_n$ acts on the multilinear components of these algebras. While $\mathfrak{S}_n$ acts trivially on the multilinear components of the colored symmetric algebra, we use poset topology techniques to understand the representation on its Koszul dual. We introduce an $\mathfrak{S}_n$-poset of weighted subsets that we call the weighted boolean algebra and we prove that the multilinear components of the colored exterior algebra are $\mathfrak{S}_n$-isomorphic to the top cohomology modules of its maximal intervals. We use a technique of Sundaram to compute group representations on Cohen–Macaulay posets to give a generating formula for the Frobenius series of the colored exterior algebra. We exploit that formula to find an explicit expression for the expansion of the corresponding representations in terms of irreducible $\mathfrak{S}_n$-representations. We show that the two colored Koszul dual algebras are Koszul in the sense of Priddy.

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1. Introduction

Let $\mathbf{k}$ denote an arbitrary field of characteristic not equal 2 and V be a finite dimensional $\mathbf{k}$-vector space. The *tensor algebra* $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ is the free associative algebra generated by V, where $V^{\otimes n}$ denotes the tensor product of n copies of V and where $V^{\otimes 0} := \mathbf{k}$. For any set $R \subseteq T(V)$ denote by $\langle R\rangle$ the ideal of $T(V)$ generated by R. Let R_1 be the subspace of $V \otimes V$ generated by the set of relations of the form

$$x \otimes y - y \otimes x \qquad \text{(symmetry)}, \tag{1.1}$$

for all $x, y \in V$. The *symmetric algebra* $\mathcal{S}(V)$ is the quotient algebra

$$\mathcal{S}(V) := T(V)/\langle R_1\rangle.$$

Now let $R_2 \subseteq V \otimes V$ be generated by the set of relations of the form

$$x \otimes y + y \otimes x \qquad \text{(antisymmetry)}, \tag{1.2}$$

for all $x, y \in V$. The *exterior algebra* $\Lambda(V)$ is the algebra

$$\Lambda(V) := T(V)/\langle R_2\rangle.$$

We will use the concatenation xy to denote the image of $x \otimes y$ in $\mathcal{S}(V)$ and the *wedge* $x \wedge y$ to denote the image of $x \otimes y$ in $\Lambda(V)$ under the canonical epimorphisms.

Let $V^* := \mathrm{Hom}(V, \mathbf{k})$ denote the vector space dual to V. For finite dimensional V we have that $V^* \simeq V$. Recall that for an associative algebra $A = A(V, R) := T(V)/\langle R\rangle$ generated on a finite dimensional vector space V and (quadratic) relations $R \subseteq V^{\otimes 2}$ there is another algebra $A^!$ associated to A that is called the *Koszul dual associative algebra* to A. Indeed, when V is finite dimensional, there is a canonical isomorphism $(V^{\otimes 2})^* \simeq V^* \otimes V^*$ and we let $R^\perp$ be the image under this isomorphism of the space of elements in $(V^{\otimes 2})^*$ that vanish on R. The *Koszul dual* $A^!$ of A is the algebra $A^! := A(V^*, R^\perp) = T(V^*)/\langle R^\perp\rangle$. It is known and easy to check from the relations (1.1) and (1.2) (see for example [13]) that $\Lambda(V^*)$ is the Koszul dual associative algebra to $\mathcal{S}(V)$.

Denote $[n] := \{1, 2, \ldots, n\}$ and let $V = \mathbf{k}\{[n]\}$ be the vector space with generators $[n]$. We define the *multilinear component* $\mathcal{S}(n)$ as the subspace of $\mathcal{S}(V)$ linearly generated by products of the form $\sigma(1)\sigma(2)\cdots\sigma(n)$ where σ is a permutation in the symmetric group $\mathfrak{S}_n$. Similarly $\Lambda(n)$ is defined to be the subspace of $\Lambda(V)$ linearly generated by *wedged permutations*, i.e., the generators are of the form $\sigma(1)\wedge\sigma(2)\wedge\cdots\wedge\sigma(n)$ for $\sigma \in \mathfrak{S}_n$. The symmetric group acts on the generators of $\mathcal{S}(n)$ and $\Lambda(n)$ by permuting their letters and this action induces representations of $\mathfrak{S}_n$ in both $\mathcal{S}(n)$ and $\Lambda(n)$. Using the relations (1.1) and (1.2) we can see that both $\mathcal{S}(n)$ and $\Lambda(n)$ are always one-dimensional spaces with bases given by $\{12\cdots n\}$ and $\{1 \wedge 2 \wedge \cdots \wedge n\}$ respectively. Moreover, for $n \geq 1$ it is easy to see that