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Rational curves on complete intersections in positive characteristic

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ABSTRACT

We study properties of rational curves on complete intersections in positive characteristic. It has long been known that in characteristic 0, smooth Calabi–Yau and general type varieties are not uniruled. In positive characteristic, however, there are well-known counterexamples to this statement. We will show that nevertheless, a *general* Calabi–Yau or general type complete intersection in projective space is not uniruled. We will also show that the space of complete intersections of degree $(d_1, \cdots, d_k)$ containing a rational curve has codimension at least $\sum_{i=1}^{k} d_i - 2n + 2$ and give similar results for hypersurfaces containing higher genus curves.

1. Introduction

Over the complex numbers, there is to some extent a dichotomy in the behavior of rational curves on smooth complete intersections in projective space. If the complete intersection is Fano, then it is rationally connected, i.e., there is a rational curve connecting any two points. On the other hand, if the complete intersection is Calabi–Yau or of general type, then it is not even uniruled, i.e., there is no rational curve passing through a very general point.

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In positive characteristic, the notions of rational connectedness and uniruledness become more complicated. While there are still notions of rational connectedness or uniruledness as above, we can alternatively require a variety to be *separably* rationally connected or uniruled, which essentially means that there are rational curves on the variety which have many infinitesimal deformations.

It is still true that all Fano varieties are rationally connected, whereas Calabi–Yau and general type varieties are never separably uniruled. More recently, it has been shown that the general Fano complete intersection is even separably rationally connected [2]. However, in positive characteristic there are general type varieties that are uniruled. Shioda constructed examples of smooth hypersurfaces of arbitrarily large degree which are unirational, so in particular, rationally connected and uniruled [19]. Liedtke shows that supersingular K3 surfaces are unirational [12], and also gives a construction of some families of uniruled surfaces of general type [13].

In this paper, we will show that despite the existence of these pathological examples, we still have the following result even in positive characteristic.

Theorem 1.1. *Let X be a general Calabi–Yau or general type complete intersection in projective space. Then X is not uniruled. Actually, more is true: there is no open subset $U \subset X$ with $\mathrm{CH}_0(U) = 0$.*

Christian Liedtke has kindly pointed out to us that one can use crystalline cohomology to prove the non-uniruledness result for surfaces, and Dingxin Zhang pointed out to us that a similar argument with crystalline cohomology can be extended to higher dimensions.

Using Theorem 1.1 together with the methods of [18], we can also obtain more quantitative results about uniruled hypersurfaces and hypersurfaces containing rational curves. In particular, we can bound the codimension of the locus of hypersurfaces containing a rational curve. When we talk about the codimension of a countable union of varieties, we mean the minimum of the codimensions of the components. For instance, it is possible to have a dense set with positive codimension. We show the following.

Theorem 1.2. *For $d \geq 2n-1$, a very general hypersurface will contain no rational curves, and moreover, the locus of hypersurfaces that contain rational curves will have codimension at least $d - 2n + 2$.*

Theorem 1.2 transports results of Ein to positive characteristic [7]. We can then use this result to deduce some results about higher genus curves too.

Theorem 1.3. *For $d \geq 2n - 1 + \lceil \frac{g-1}{2} \rceil$, the very general hypersurface of degree d in $\mathbb{P}^n$ does not contain any curve of geometric genus g.*

Clemens gives better bounds in characteristic 0 (see [3]), but it remains open whether his bounds hold in positive characteristic.