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Coordinates at stable points of the space of arcs

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A B S T R A C T

Let X be a variety over a field k and let X_∞ be its space of arcs. Let P_E be the stable point of X_∞ defined by a divisorial valuation ν_E on X. Assuming char $k = 0$, if X is smooth at the center of P_E, we make a study of the graded algebra associated to ν_E and define a finite set whose elements generate a localization of the graded algebra modulo étale covering. This provides an explicit description of a minimal system of generators of the local ring $\mathcal{O}_{X_\infty, P_E}$. If X is singular, we obtain generators of $P_E \, / \, P_E^2$ and conclude that embdim $\mathcal{O}_{(X_\infty)_{\mathrm{red}}, P_E} =$ embdim $\widehat{\mathcal{O}_{X_\infty, P_E}} \leq \widehat{k}_E + 1$ where $\widehat{k}_E$ is the Mather discrepancy of X with respect to ν_E. This provides algebraic tools for explicit computations of the local rings $\widehat{\mathcal{O}_{X_\infty, P_E}}$.

1. Introduction

The space of arcs X_∞ of an algebraic variety X was introduced by J. Nash in the 60's [20]. He expected to detect from arc families those components of the exceptional locus of the resolutions of singularities $Y \to X$ which are invariant by birational equivalence. The space X_∞ is an intrinsic object associated to X which allows to construct invariants of

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the variety: J. Denef and F. Loeser [4] made a systematic construction of some invariants of X using motivic integration on X_∞. They also considered X_∞ together with its scheme structure, an idea which was further developed by S. Ishii and J. Kollar [13].

Precisely, J. Nash asked "how complete is the representation of essential components by arc families", that is, what essential valuations are determined by the irreducible components of the space of arcs X_∞^{Sing} centered at some singular point of X. Divisorial valuations whose center appears as an irreducible component of the exceptional locus of every resolution of singularities of X are called essential valuations.

The arc families considered by J. Nash correspond to certain fat points P of X_∞. These fat points are *stable points*, as defined by the author in [23] lemma 3.1 and [24] definition 3.1 (see also [25]). It is natural to expect that those geometric properties of X_∞ with respect to an arc family are reflected in the algebraic properties of the local ring $\mathcal{O}_{X_\infty,P}$. An important role is played by the following algebraic property: the ideal of definition of a stable point P of X_∞ in a neighborhood of P, with its reduced structure, is finitely generated ([24] theorem 4.1, see 2.4 below). This implies that the complete local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ is a Noetherian ring ([24], corollary 4.6).

Most advances on the Nash program on arc families use our Curve Seletion Lemma ([24] corollary 4.8) which is an easy consequence of the previous property, and is valid over a perfect field of any characteristic. For instance, if $\dim \mathcal{O}_{X_\infty,P} = 1$ then P is the generic point of an irreducible component of X_∞^{Sing} ([25] corollary 5.12). This is what occurs for essential valuations in toric varieties ([13] theorem 3.16), for nonuniruled ([16] theorem 3.3) and for terminal valuations ([8] theorem 3.3). Known counterexamples involve a local ring $\widehat{\mathcal{O}_{X_\infty,P}}$ of dimension greater than or equal to 2: for the counterexample in [13] see [25] remark 5.16, and for the ones in [14] see example 4.13 below. On the other hand, essential valuations are characterized in different ways: minimal elements in the cone for toric varieties ([3] theorem 1.10 and [13] section 3), nonruled ([1] proposition 4) and some divisorial valuations with discrepancy 1 (resp. 2) over certain canonical (resp. terminal) isolated singularities ([7] lemma 5.2 and [14]).

The local rings $\widehat{\mathcal{O}_{X_\infty,P}}$ involved in these results and examples have nevertheless a simple algebraic structure: nonreduced curves and some mildly singular surfaces. But in general their structure is much more complicated. Our purpose in this article is to develop appropriate algebraic tools for computing the local rings $\widehat{\mathcal{O}_{X_\infty,P}}$, P a stable point. We construct (Corollary 4.11) a presentation of $\widehat{\mathcal{O}_{X_\infty,P}}$ by concrete generators and relations:

$$\widehat{\mathcal{O}_{X_\infty,P}} \;\cong\; \kappa(P)\left[\left[\{X_{j,r;n}\}_{(j,r;n)\in\mathcal{C}}\right]\right] \Big/ \;\; \widetilde{I}$$

where $P = P_{eE}$ is the generic point of the family of arcs with contact $e \geq 1$ with an exceptional divisor E and the cardinal of $\mathcal{C}$ is $e(\widehat{k}_E + 1)$. Here $\widehat{k}_E$ is the Mather discrepancy of X with respect to E (see [12], [5]). This provides a framework to recover geometric properties and invariants of a variety X. We apply our result in Corollary 4.11 to understand the geometry of the space of arcs in the examples of [14] (Example 4.13).