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# Correspondence functors and finiteness conditions

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ABSTRACT

We investigate the representation theory of finite sets. The correspondence functors are the functors from the category of finite sets and correspondences to the category of $k$-modules, where $k$ is a commutative ring. They have various specific properties which do not hold for other types of functors. In particular, if $k$ is a field and if $F$ is a correspondence functor, then $F$ is finitely generated if and only if the dimension of $F(X)$ grows exponentially in terms of the cardinality of the finite set $X$. Moreover, in such a case, $F$ has actually finite length. Also, if $k$ is noetherian, then any subfunctor of a finitely generated functor is finitely generated.



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## 1. Introduction

Representations of categories have been used by many authors in different contexts. The present paper is the first in a series which develops the theory in the case of the category whose objects are all finite sets and morphisms are all correspondences between finite sets.

For representing a category of finite sets, there are several possible choices. Pirashvili [16] treats the case of pointed sets and maps, while Church, Ellenberg and Farb [9] consider the case where the morphisms are all injective maps. Putman and Sam [17] use all $k$-linear splittable injections between finite-rank free $k$-modules (where $k$ is a commutative ring). Here, we move away from such choices by using all correspondences as morphisms. The cited papers are concerned with applications to cohomological stability, while we develop our theory without any specific application in mind. The main motivation is provided by the fact that finite sets are basic objects in mathematics. Moreover, the theory turns out to have many quite surprising results, which justify the development presented here.

Let $\mathcal{C}$ be the category of finite sets and correspondences. We define a *correspondence functor* over a commutative ring $k$ to be a functor from $\mathcal{C}$ to the category $k$-Mod of all $k$-modules. As much as possible, we develop the theory for an arbitrary commutative ring $k$. However, let us start with the case when $k$ is a field. If $F$ is a correspondence functor over a field $k$, we prove that $F$ is finitely generated if and only if the dimension of $F(X)$ grows exponentially in terms of the cardinality of the finite set $X$ (Theorem 8.7). In such a case, we also prove the striking fact that $F$ has finite length (Theorem 9.2). This result was obtained independently by Gitlin [11] (for a field $k$ of characteristic zero, or algebraically closed), using a criterion proved by Wiltshire-Gordon [20]. Moreover, for finitely generated correspondence functors, we show that the Krull–Remak–Schmidt theorem holds (Proposition 6.6) and that projective functors coincide with injective functors (Theorem 10.6).

Suppose that $k$ is a field. By well-known results about representations of categories, simple correspondence functors can be classified. In our case, they are parametrized by triples $(E, R, V)$, where $E$ is a finite set, $R$ is a partial order relation on $E$, and $V$ is a simple $k\,\mathrm{Aut}(E, R)$-module (Theorem 4.7). This is the first indication of the importance of posets in our work. However, if $S_{E,R,V}$ is the simple functor parametrized by $(E, R, V)$, then it is quite hard to describe the evaluation $S_{E,R,V}(X)$ at a finite set $X$. We will achieve this in a future paper [8] by giving a closed formula for its dimension.

A natural question when dealing with a commutative ring $k$ is to obtain specific results when $k$ is noetherian. We follow this track in Section 11 and show for instance that any subfunctor of a finitely generated correspondence functor is again finitely generated (Corollary 11.5). Also, we obtain stabilization results for Hom and Ext between correspondence functors evaluated at large enough finite sets (Theorem 12.3).

This article uses essentially only standard facts from algebra and representation theory, with the following exceptions. A few basic results in Section 2 have been imported