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Generalised quadratic forms and the u-invariant

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ABSTRACT

The u-invariant of a field is the supremum of the dimensions of anisotropic quadratic forms over the field. We define corresponding u-invariants for hermitian and generalised quadratic forms over a division algebra with involution in characteristic 2 and investigate the relationships between them. We investigate these invariants in particular in the case of a quaternion algebra and further when this quaternion algebra is the unique quaternion division algebra over a field.

1. Introduction

The u-invariant is a classical invariant of a field. For a field F, the invariant $u(F)$ is defined as the supremum of the dimensions of anisotropic quadratic forms over F.

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The u-invariant is a much studied topic, and results include the celebrated proof of Merkurjev in [21] that for every even number there exists a field with u-invariant of that value, disproving a long standing conjecture of Kaplansky that the u-invariant must be a 2-power or infinite. See [13] for a survey of this and related questions.

The u-invariants of fields of characteristic 2 have also been studied. It was noted in [2] that it is convenient to consider two different u-invariants, one related to anisotropic quadratic forms in general, and one related to the anisotropy of nonsingular quadratic forms. In characteristic different from 2 these invariants are equal, as any singular quadratic forms over such fields are isotropic. In characteristic 2, however, they can hold very different values, see for example [20]. In characteristic 2, one can also consider similar invariants related to the anisotropy of symmetric bilinear forms, rather than quadratic forms. Again, this invariant is equal to the usual u-invariant for nonsingular quadratic forms in characteristic different from 2, as here symmetric bilinear forms and quadratic forms are equivalent objects.

In [19] a hermitian version of the u-invariant was introduced in characteristic different from 2. If (D, θ) is a division algebra over F with involution θ, then $u(D, \theta)$ is the supremum of the dimensions of anisotropic hermitian forms over (D, θ). This definition can also be applied in characteristic 2. However, as for fields, there is a large variety of invariants analogous to this u-invariant to investigate in characteristic 2. These include other invariants related to hermitian forms, but also invariants related to generalised quadratic forms. Generalised quadratic forms (also known as pseudo-quadratic forms) are an extension of the concept of quadratic forms over a field to the setting of central simple division algebras with involution, first introduced in [29]. They generalise quadratic forms to this setting in an analogous manner to the generalisation of symmetric bilinear forms to hermitian forms (see Section 3). In particular, they are used to study twisted orthogonal groups in characteristic 2 much as hermitian forms are used to study twisted orthogonal groups in characteristic different from 2. So far the study of generalised quadratic forms has been largely restricted to nonsingular quadratic forms, with singular forms mainly only being considered over fields (see, for example, [15]). Here we study u-invariants related both to nonsingular generalised quadratic forms as well as to forms that may be singular, following on from work begun in [9]. It is also convenient to consider u-invariants related to hermitian forms of specific types, specifically alternating and direct hermitian forms (the latter were introduced in [6]).

After some general preliminary results, we introduce various invariants for generalised quadratic and hermitian forms in characteristic 2 in Section 4 and investigate their properties and interrelations. In Section 5 we calculate our u-invariant related to direct hermitian forms for division algebras that are a tensor product of quaternion algebras, generalising a result for symmetric bilinear forms over fields (see Remark 4.5). We consider the values of our invariants for F-quaternion algebras in Section 6 and determine them when there is a unique square class in F. We further investigate our invariants in the case where the given quaternion algebra is the unique division algebra