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Power associative nilalgebras of dimension 9

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A B S T R A C T

In [19] authors described some properties about commutative power associative nilalgebras of nilindex 5. Here we will get new results about the structure of this class of algebras. Those results will allow us to prove that every commutative power associative algebra of dimension 9 and nilindex 5 over a field of characteristic different from 2, 3 and 5 is solvable. Consequently the famous Albert's conjecture ([1], Problem 1.1) is setted for dimension ≤ 9 and characteristic 0 since the case of dimension 9 and nilindex $\neq 5$ have already been examined in [27], [11], [14] and [20].

1. Preliminaries

Let $\mathfrak{A}$ be a commutative algebra (not necessary associative) over a field F. There are different ways to define the powers of an element in $\mathfrak{A}$. For $a \in \mathfrak{A}$, its *principal powers* are defined inductively as $a^1 = a$ and $a^{k+1} = a^k a$ for any $k > 1$. An element $a \in \mathfrak{A}$ is *(principal) nilpotent* if there exists a positive integer t such that $a^t = 0$. An algebra consisting only of nilpotent elements is called *(principal) nilalgebra*. If there exists a positive integer t such that $a^t = 0$ for any $a \in \mathfrak{A}$, then the smallest positive integer with this property is called *(principal) nilindex* of the algebra or index of the nilalgebra.

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Let B and C be linear subspace of an algebra $\mathfrak{A}$ over the field F. The product BC is the linear subset of $\mathfrak{A}$ gives by $\{bc: \ b \in B, \ c \in C\}$. A *nilpotent algebra* is an algebra for which there is a natural number t such that any product containing at least t elements of the algebra is zero. Thus, an algebra $\mathfrak{A}$ is nilpotent of index $\leq t$ if we have the identity $\mathfrak{A}^t = \{0\}$, where the powers of the algebra $\mathfrak{A}$ are defined as follows: $\mathfrak{A}^1 = \mathfrak{A}$ and $\mathfrak{A}^k = \sum_{i+j=k} \mathfrak{A}^i\mathfrak{A}^j$ for all $k \geq 2$. The algebra $\mathfrak{A}$ is *solvable* of index less than or equal to t if the tth plenary power of $\mathfrak{A}$ vanishes, $\mathfrak{A}^{(t)} = \{0\}$, where the *plenary powers* of $\mathfrak{A}$ are defined inductively as follows: $\mathfrak{A}^{(0)} = \mathfrak{A}$ and $\mathfrak{A}^{(k)} = \mathfrak{A}^{(k-1)}\mathfrak{A}^{(k-1)}$ for all positive integers k.

An algebra $\mathfrak{A}$ is called *power associative* if for any $a \in \mathfrak{A}$, the subalgebra $F[a]$ of $\mathfrak{A}$ generated by the element a is associative. Thus, an algebra is power associative if and only if $a^ia^j = a^{i+j}$ for all $a \in \mathfrak{A}$ and all positive integers i, j. In a series of papers [2–4] A.A. Albert studied the structure of this class of algebras. In particular, the article [2] proved the following result.

Theorem 1. *Let $\mathfrak{A}$ be a commutative algebra over a field F whose characteristic is either 0 or prime with 30. Then $\mathfrak{A}$ is a power associative algebra if and only if we have that $x^2x^2 = x^4$ for any $x \in \mathfrak{A}$.*

For associative algebras over a field of characteristics zero or greater than t, Dubnov–Ivanov–Nagata–Higman Theorem affirms that every associative nilalgebra of nilindex t over an infinite field is nilpotent of bounded index (see [31] and [26]). The analogous question for nearly associative algebras has been considered by many authors. In contrast to the associative case, G.V. Dorofeev [9] constructed an example of a solvable nonnilpotent alternative algebra (see also [31], p. 127). Thus, the Dubnov–Ivanov–Nagata–Higman theorem cannot be generalized to arbitrary alternative, Jordan or power associative algebras. For Jordan algebras, A.A. Albert shows that any finite-dimensional Jordan nilalgebra of characteristic different from 2 is nilpotent, and E. Zelmanov [32] states that every Jordan nilalgebra of bounded nilindex over a field of characteristic different from 2 is locally nilpotent. In 1962, K.A. Zhevlakov (see [31]) proved that an alternative nilalgebra of index t over a field of characteristic either 0 or greater than t is solvable. We also point out the remarkable theorem of E. Zelmanov and V.G. Skosyrskii [27] which states that a special Jordan nilalgebra of index t with characteristic zero or greater than $2t$ is solvable. Furthermore, it is proved that every finitely generated commutative algebra with the Engel identity $x(x(xy)) = 0$ is solvable [5] and every finite-dimensional commutative algebra with the same identity is nilpotent [15].

For power associative algebras, Gerstenhaber and Myung [25] proved that every commutative power associative nilalgebra of dimension ≤ 4 is nilpotent. In 1972, D. Suttles [28] gave an example of a commutative power associative algebra of dimension 5 and nilindex 4 which is not nilpotent, but solvable. Thus, a modified version of Albert's problem was formulated in the Dnestr Notebook [1], Problem 1.1.