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A short note about diffuse Bieberbach groups

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ABSTRACT

We consider low dimensional diffuse Bieberbach groups. In particular we classify diffuse Bieberbach groups up to dimension 6. We also answer a question from [7, page 887] about the minimal dimension of a non-diffuse Bieberbach group which does not contain the three-dimensional Hantzsche–Wendt group.

1. Introduction

The class of diffuse groups was introduced by B. Bowditch in [2]. By definition a group Γ is *diffuse*, if every finite non-empty subset $A \subset \Gamma$ has an extremal point, i.e. an element $a \in A$ such that for any $g \in \Gamma \setminus \{1\}$ either ga or $g^{-1}a$ is not in A. Equivalently (see [7]) a group Γ is diffuse if it does not contain a non-empty finite set without extremal points.

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The interest in diffuse groups follows from Bowditch's observation that they have the unique product property.[1]

Originally unique products were introduced in the study of group rings of discrete, torsion-free groups. More precisely, it is easily seen that if a group Γ has the unique product property, then it satisfies Kaplansky's unit conjecture. In simple terms this means that the units in the group ring $\mathbb{C}[\Gamma]$ are all trivial, i.e. of the form λg with $\lambda \in \mathbb{C}^*$ and $g \in \Gamma$. For more information about these objects we refer the reader to [1], [9, Chapter 10] and [7]. In part 3 of [7] the authors prove that any torsion-free crystallographic group (Bieberbach group) with trivial center is not diffuse. By definition a crystallographic group is a discrete and cocompact subgroup of the group $O(n) \ltimes \mathbb{R}^n$ of isometries of the Euclidean space $\mathbb{R}^n$. From Bieberbach's theorem (see [12]) the normal subgroup T of all translations of any crystallographic group Γ is a free abelian group of finite rank and the quotient group (holonomy group) $\Gamma/T = G$ is finite.

In [7, Theorem 3.5] it is proved that for a finite group G:

1. If G is not solvable then any Bieberbach group with holonomy group isomorphic to G *is not* diffuse.
2. If every Sylow subgroup of G is cyclic then any Bieberbach group with holonomy group isomorphic to G *is* diffuse.
3. If G is solvable and has a non-cyclic Sylow subgroup then there are examples of Bieberbach groups with holonomy group isomorphic to G which *are* and examples which *are not* diffuse.

Using the above the authors of [7] classify non-diffuse Bieberbach groups in dimensions ≤ 4. One of the most important non-diffuse groups is the 3-dimensional Hantzsche–Wendt group, denoted in [11] by Δ_P. For the following presentation

$$\Delta_P = \langle x, y \mid x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2} \rangle$$

the maximal abelian normal subgroup is generated by x^2, y^2 and $(xy)^2$ (see [6, page 154]). At the end of part 3.4 of [7] the authors ask the following question.

Question 1. What is the smallest dimension d_0 of a non-diffuse Bieberbach group which does not contain Δ_P?

The answer for the above question was the main motivation for us. In fact we prove, in the next section, that $d_0 = 5$. Moreover, we extend the results of part 3.4 of [7] and with support of computer, we present the classification of all Bieberbach groups in dimension $d \leq 6$ which are (non)diffuse.

[1] The group Γ is said to have the unique product property if for every two finite non-empty subsets $A, B \subset \Gamma$ there is an element in the product $x \in A\dot{B}$ which can be written uniquely in the form $x = ab$ with $a \in A$ and $b \in B$.

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