Journal of Algebra 472 (2017) 273–278

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The existence of a polynomial factorization map for some compact linear groups

O.G. Styrt

ARTICLE INFO

Article history:
Received 21 November 2014
Available online 11 November 2016
Communicated by Vera Serganova

MSC:
14L24
22C05

Keywords:
Lie group
Factorization map of an action

ABSTRACT

It is proved that each of compact linear groups of one special type admits a polynomial factorization map onto a real vector space. More exactly, the group is supposed to be non-commutative one-dimensional and to have two connected components, and its representation should be the direct sum of three irreducible two-dimensional real representations at least two of them being faithful.

1. Introduction

For a group G and a set M, we will write «action $G\colon M$» for «action of G on M» and call a surjective map from M to some set *a factorization map* of such an action if its fibres coincide with G-orbits. For a Lie group G, we will denote by G^0 its connected component. Let $\mathbb{T}$ be the multiplicative Lie group $\{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$.

In the paper, we prove that each of the compact linear groups of one certain type admits a polynomial factorization map onto a vector space.

This problem arose from the question when the topological quotient of a compact linear group is homeomorphic to a vector space that was researched in [1–5]. In [1], the result for finite groups is obtained. Namely, the necessary and sufficient condition is

E-mail address: oleg_styrt@mail.ru.

http://dx.doi.org/10.1016/j.jalgebra.2016.10.020
0021-8693/

that the group is generated by pseudoreflections. Moreover, for the most of such groups, a polynomial factorization map onto a vector space is explicitly constructed. In [2], the case of infinite groups with commutative connected components is considered. For any $n_1, n_2, n_3 \in \mathbb{N}$, denote by $G(n_1, n_2, n_3)$ the subgroup of the group $\mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$ generated by the operators $(z_1, z_2, z_3) \to (\lambda^{n_1} z_1, \lambda^{n_2} z_2, \lambda^{n_3} z_3)$ $(\lambda \in \mathbb{T})$ and $(z_1, z_2, z_3) \to (\overline{z}_1, \overline{z}_2, \overline{z}_3)$. In [2], it is proved that $\mathbb{C}^3 \,/\, \big(G(n_1, n_2, n_3)\big) \cong \mathbb{R}^5$ for all $n_1, n_2, n_3 \in \mathbb{N}$. However, the proof is based on purely topological aspects. Unlikely, for one of the main examples considered in [2], namely, for the linear group $\Big\{\mathrm{diag}(\lambda_1, \ldots, \lambda_n)\colon \lambda_1, \ldots, \lambda_n \in \mathbb{T}, \prod_{i=1}^{n} \lambda_i^{a_i} = 1\Big\} \subset \mathbf{GL}(\mathbb{C}^n)$ $(n, a_1, \ldots, a_n \in \mathbb{N})$, an $\mathbb{R}$-polynomial factorization map onto a vector space is given explicitly. In this work, we prove that each of the linear groups $G(1, 1, n) \subset \mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$ $(n \in \mathbb{N})$ admits an $\mathbb{R}$-polynomial factorization map $\mathbb{C}^3 \twoheadrightarrow \mathbb{R}^5$.

Fix an arbitrary number $n \in \mathbb{N}$. Set $G := G(1, 1, n) \subset \mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$.

The group $G \subset \mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$ is generated by the operators $(z_1, z_2, z_3) \to (\lambda z_1, \lambda z_2, \lambda^n z_3)$ $(\lambda \in \mathbb{T})$ and $(z_1, z_2, z_3) \to (\overline{z}_1, \overline{z}_2, \overline{z}_3)$. The map $\mathbb{C}^3 \to \mathbb{C}^3$, $z \to w$, where $w_1 := z_1 + z_2$, $w_2 := \overline{(z_1 - z_2)} = \overline{z}_1 + \overline{z}_2$, $w_3 := z_3$, is an $\mathbb{R}$-linear automorphism. In terms of the so-defined coordinates w_1, w_2, w_3, the linear group $G \subset \mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$ is generated by the operators $(w_1, w_2, w_3) \to (\lambda w_1, \overline{\lambda} w_2, \lambda^n w_3)$ $(\lambda \in \mathbb{T})$ and $\tau\colon (w_1, w_2, w_3) \to (w_2, w_1, \overline{w}_3)$. The group $G^0 \subset \mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$ consists of all the operators $(w_1, w_2, w_3) \to (\lambda w_1, \overline{\lambda} w_2, \lambda^n w_3)$ $(\lambda \in \mathbb{T})$. Also, $G = G^0 \sqcup (G^0 \tau)$ and $G/G^0 \cong \mathbb{Z}_2$.

In this paper, we will show that the (obviously, $\mathbb{R}$-polynomial) map

$$\mathbb{C}^3 \to \mathbb{R}^5 \cong \mathbb{C}^2 \oplus \mathbb{R},\ (w_1, w_2, w_3) \to \Big(w_2^n w_3 + w_1^n \overline{w}_3, w_1 w_2, |w_3|^2 - \big(|w_1|^{2n} + |w_2|^{2n}\big)\Big)$$

is a factorization map of the linear group $G \subset \mathbf{GL}_{\mathbb{R}}(\mathbb{C}^3)$.

2. Proof of the main result

Denote by Γ the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, by $\gamma_0, \gamma', \gamma''$ its non-trivial elements, and by Γ_0 the subgroup $\{e, \gamma_0\} \subset \Gamma$. Consider the set $C := \mathbb{R}^3_{\geqslant 0} \times \mathbb{C}^3$ and the action $\Gamma\colon C$ uniquely defined by the relations $\gamma' p := (r_2, r_1, r_3, v_1, v_2, v_3)$ and $\gamma'' p := (r_1, r_2, r_3, v_2, v_1, v_3)$ $(p = (r_1, r_2, r_3, v_1, v_2, v_3) \in C)$. Since $G/G^0 \cong \mathbb{Z}_2 \cong \Gamma_0$, there exists a homomorphism $\rho\colon G \twoheadrightarrow \Gamma_0$ such that $\operatorname{Ker} \rho = G^0$ and $\rho(\tau) = \gamma_0$. The subset $M \subset C$ of all elements $(r_1, r_2, r_3, v_1, v_2, v_3) \in C$ such that $|v_1| = r_2^n r_3$, $|v_2| = r_1^n r_3$, $|v_3| = r_1 r_2$, $v_1 v_2 = r_3^2 v_3^n$ is Γ_0-invariant.

The map $\pi_0\colon \mathbb{C}^3 \to C$, $(w_1, w_2, w_3) \to \big(|w_1|, |w_2|, |w_3|, w_2^n w_3, w_1^n \overline{w}_3, w_1 w_2\big)$ satisfies $\pi_0(\mathbb{C}^3) = M \subset C$, and its fibres coincide with the orbits of the action $G^0\colon \mathbb{C}^3$. It is easy to see that $\pi_0 \circ \tau \equiv \gamma_0 \circ \pi_0$. Thus,

$$\begin{aligned} &\forall\, g \in G && \pi_0 \circ g \equiv \rho(g) \circ \pi_0; \\ &\forall\, x, y \in \mathbb{C}^3 && (y \in Gx) \quad \Leftrightarrow \quad \Big(\pi_0(y) \in \Gamma_0\big(\pi_0(x)\big)\Big). \end{aligned} \tag{2.1}$$