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Torification of diagonalizable group actions on toroidal schemes ☆

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A B S T R A C T

We study actions of diagonalizable groups on toroidal schemes (i.e. logarithmically regular logarithmic schemes). In particular, we show that for so-called toroidal actions the quotient is again a toroidal scheme. Our main result constructs for an arbitrary action a canonical torification by an equivariant blowings up. This extends earlier results of Abramovich–de Jong, Abramovich–Karu–Matsuki–Włodarczyk, and Gabber in various aspects.

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1. Introduction

1.1. Toroidal actions, quotients and torification

Consider a variety X with toroidal structure and an action of a group G on X. *Torification* is a blowing-up process $X' \to X$ which guarantees that the quotient map $X' \to X' /\!\!/ G$ is toroidal. It was introduced in [1] when G is finite for the purpose of proving resolution of singularities; and in [3] when $G = \mathbb{G}_m$ for proving factorization of birational maps.

In this paper we consider G diagonalizable, and prove a general torification result for arbitrary toroidal schemes (see Section 2.3), not necessarily over a field:

Theorem 1.1.1 *(See Theorem 4.6.5). Assume that a diagonalizable group G acts in a relatively affine manner on a toroidal scheme (X, D). Then there is a G-equivariant modification $F_{(X,D)} \colon X' \to X$, such that, denoting by D' be the union of the preimage of D and the exceptional divisor of $F_{(X,D)}$, we have*

(i) The pair (X', D') is toroidal and the natural G-action on (X', D') is toroidal.

(ii) The morphism $F_{(X,D)}$ is functorial with respect to surjective strongly equivariant strict morphisms $h\colon (Y, E) \to (X, D)$ of toroidal schemes in the sense that $F_{(Y,E)}$ is the base change of $F_{(X,D)}$.

We refer to Section 1.3 and [5, Section 5.3.1] for the notions of relatively affine actions and strongly equivariant morphisms, and to Section 2.3.16 for the notion of morphisms between toroidal schemes. A slightly more precise statement of Theorem 4.6.5 in terms of blowing up is provided in Theorem 5.4.5.

Theorem 1.1.1 builds on a more detailed Theorem 4.5.1 which assumes the action to be G-simple, see Section 3.1.4. We further optimize that result as follows:

Theorem 1.1.2 *(See Theorem 5.4.2). Assume that a toroidal scheme (X, D) is provided with a relatively affine,* G-simple *action of a diagonalizable group $G = \mathbf{D}_L$. Assume X contains a strongly equivariant dense open set U on which G acts freely. There exist ideal sheaves I_X and $\tilde{I}_X$ on X and $\widetilde{X} = X /\!\!/ G$ with resulting blowings up $f_{(X,D)} \colon X' \to X$ and $\widetilde{f}_{(X,D)} \colon \widetilde{X}' \to \widetilde{X}$, such that I_X is locally generated by G-invariants, and, denoting by D' the union of the preimage of D and the exceptional divisor of $f_{(X,D)}$, we have*

(i) The pair (X', D') is toroidal and the natural G-action on (X', D') is toroidal.

(ii) The morphism of quotients $f_{(X,D)} /\!\!/ G : X' /\!\!/ G \to \widetilde{X}$ is $\widetilde{f}_{(X,D)}$.

(iii) The blowings up $f_{(X,D)}$ and $\widetilde{f}_{(X,D)}$ are functorial with respect to surjective strongly equivariant strict morphisms $h\colon (Y, E) \to (X, D)$ of toroidal schemes: denoting $\widetilde{h} = h /\!\!/ G$, we have $h^*(I_X) = I_Y$, $f^{\mathrm{tor}}_{(Y,E)} = f^{\mathrm{tor}}_{(X,D)} \times_X Y$, $\widetilde{h}^*(\tilde{I}_X) = \tilde{I}_Y$, and $\widetilde{f}_{(Y,E)} = \widetilde{f}_{(X,D)} \times_{\widetilde{X}} \widetilde{Y}$.

(iv) If $V \subseteq X$ is a strongly equivariant open subset such that the action on $(V, D|_V)$ is toroidal then I_X restricts to the unit ideal on V and $V \times_X X' = V$.