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On centers of blocks with one simple module

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Let G be a finite group, and let B be a non-nilpotent block of G with respect to an algebraically closed field of characteristic 2. Suppose that B has an elementary abelian defect group of order 16 and only one simple module. The main result of this paper describes the algebra structure of the center of B. This is motivated by a similar analysis of a certain 3-block of defect 2 in Kessar (2012) [15].

1. Introduction

This paper is concerned with the algebra structure of the center of a p-block B of a finite group G. In order to make statements precise let $(K, \mathcal{O}, F)$ be a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring of characteristic 0, K is the field of fractions of $\mathcal{O}$, and $F = \mathcal{O}/\operatorname{J}(\mathcal{O}) = \mathcal{O}/(\pi)$ is an algebraically closed field of prime characteristic p. As usual, we assume that K is a splitting field for G.

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A well-known result by Broué–Puig [8] asserts that if B is nilpotent, then the number of irreducible Brauer characters in B equals $l(B) = 1$. Since the algebra structure of nilpotent blocks is well understood by work of Puig [26], it is natural to study non-nilpotent blocks with only one irreducible Brauer character. These blocks are necessarily non-principal by Brauer's result (see [24, Corollary 6.13]) and maybe the first example was given by Kiyota [17]. Here, $p = 3$ and B has an elementary abelian defect group of order 9. More generally, a theorem by Puig–Watanabe [28] states that if the defect group of B is abelian, then B has a Brauer correspondent with more than one simple module. Ten years later, Benson–Green [2] and others [13,16] have developed a general theory of these blocks by making use of quantum complete intersections. Applying this machinery, Kessar [15] was able to describe the algebra structure of Kiyota's example explicitly. Her arguments were simplified recently in [21]. We also mention two more recent papers dealing with these blocks. Malle–Navarro–Späth [23] have shown that the unique irreducible Brauer character in B is the restriction of an ordinary irreducible character. Finally, Benson–Kessar–Linckelmann [3] studied Hochschild cohomology in order to obtain results on blocks of defect 2 with only one irreducible Brauer character.

In the present paper we deal with the second smallest example in terms of defect groups. Here, $p = 2$ and B has elementary abelian defect group D of order 16. In [22] the numerical invariants of B have been determined. In particular, it is known that the number of irreducible ordinary characters (of height 0) of B is $k(B) = k_0(B) = 8$. Moreover, the inertial quotient $I(B)$ of B is elementary abelian of order 9. Examples for B are given by the non-principal blocks of $G = \texttt{SmallGroup}(432, 526) \cong D \rtimes 3_+^{1+2}$ where 3_+^{1+2} denotes the extraspecial group of order 27 and exponent 3. Here, the center of 3_+^{1+2} acts trivially on D and $G/\operatorname{Z}(G) \cong A_4 \times A_4$ where A_4 is the alternating group of degree 4. Since the algebra structure of B seems too difficult to describe at the moment, we are content with studying the center $\operatorname{Z}(B)$ as an algebra over F. As a consequence of Broué's Abelian Defect Group Conjecture, the isomorphism type of $\operatorname{Z}(B)$ should be independent of G. In fact, our main theorem is the following.

Theorem 1.1. *Let B be a non-nilpotent 2-block with elementary abelian defect group of order 16 and only one irreducible Brauer character. Then*

$$\operatorname{Z}(B) \cong F[X, Y, Z_1, \ldots, Z_4]/\langle X^2 + 1,\ Y^2 + 1,\ (X + 1)Z_i,\ (Y + 1)Z_i,\ Z_iZ_j\rangle.$$

In particular, $\operatorname{Z}(B)$ has Loewy length 3.

The paper is organized as follows. In the second section we consider the generalized decomposition matrix Q of B. Up to certain choices there are essentially three different possibilities for Q. A result by Puig [27] (cf. [9, Theorem 5.1]) describes the isomorphism type of $\operatorname{Z}(B)$ (regarded over $\mathcal{O}$) in terms of Q. In this way we prove that there are at most two isomorphism types for $\operatorname{Z}(B)$. In the two subsequent sections we apply ring-theoretical arguments to the basic algebra of B in order to exclude one possibility for $\operatorname{Z}(B)$. Finally,