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The chain equivalence of totally decomposable orthogonal involutions in characteristic two

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ABSTRACT

It is shown that two totally decomposable algebras with involution of orthogonal type over a field of characteristic two are isomorphic if and only if they are chain equivalent.

1. Introduction

The chain equivalence theorem for bilinear Pfister forms states that two n-fold Pfister forms $\mathfrak{b}$ and $\mathfrak{b}'$ are isometric if and only if there exists a finite chain of n-fold Pfister forms starting with $\mathfrak{b}$ and ending with $\mathfrak{b}'$ such that for any two consecutive forms $\langle\langle \alpha_1, \cdots, \alpha_n \rangle\rangle$ and $\langle\langle \beta_1, \cdots, \beta_n \rangle\rangle$ in this chain, there are two indices i, j for which $\langle\langle \alpha_i, \alpha_j \rangle\rangle \simeq \langle\langle \beta_i, \beta_j \rangle\rangle$ and $\alpha_k = \beta_k$ for $k \neq i, j$ (see [6, (3.2)] and [1, (A.1)]). There exist some related results in the literature for certain classes of central simple algebras over a field. In [11], the chain equivalence theorem for biquaternion algebras over a field of characteristic not two was proved (see [2] for the corresponding result in characteristic two). Also, the chain equivalence theorem for tensor products of quaternion algebras over a field of arbitrary

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characteristic with no nontrivial quadratic 3-fold Pfister forms was recently obtained in [3].

Let F be a field of characteristic 2. An algebra with involution (A, σ) over F is called *totally decomposable* if it decomposes as tensor products of quaternion F-algebras with involution. In [4], a bilinear Pfister form $\mathfrak{Pf}(A, \sigma)$, called the *Pfister invariant*, was associated to every totally decomposable algebra with orthogonal involution (A, σ) over F. In [9, (6.5)], it was shown that the Pfister invariant can be used to classify totally decomposable algebras with orthogonal involution over F. Regarding this result, an analogous chain equivalence for these algebras was defined in [9, (6.7)]. A relevant problem then is whether the isomorphism of such algebras with involution implies that they are chain equivalent (see [9, (6.8)]). In this work we present a solution to this problem.

2. Preliminaries

In this paper, F is a field of characteristic 2.

Let V be a finite dimensional vector space over F. A symmetric bilinear form $\mathfrak{b} : V \times V \to F$ is called *anisotropic* if $\mathfrak{b}(v, v) \neq 0$ for every nonzero vector $v \in V$. The form $\mathfrak{b}$ is called *metabolic* if V has a subspace W with $\dim W = \frac{1}{2} \dim V$ and $\mathfrak{b}|_{W \times W} = 0$. For $\lambda_1, \cdots, \lambda_n \in F^\times$, the form $\langle\langle \lambda_1, \cdots, \lambda_n \rangle\rangle := \bigotimes_{i=1}^n \langle 1, \lambda_i \rangle$ is called a *bilinear Pfister form*, where $\langle 1, \lambda_i \rangle$ is the diagonal form $\mathfrak{b}((x_1, x_2), (y_1, y_2)) = x_1 y_1 + \lambda_i x_2 y_2$. By [5, (6.3)], a bilinear Pfister form is either metabolic or anisotropic. We say that $\mathfrak{b} = \langle\langle \alpha_1, \cdots, \alpha_n \rangle\rangle$ and $\mathfrak{b}' = \langle\langle \beta_1, \cdots, \beta_n \rangle\rangle$ are *simply P-equivalent*, if either $n = 1$ and $\alpha_1 F^{\times 2} = \beta_1 F^{\times 2}$ or $n \geqslant 2$ and there exist $1 \leqslant i < j \leqslant n$ such that $\langle\langle \alpha_i, \alpha_j \rangle\rangle \simeq \langle\langle \beta_i, \beta_j \rangle\rangle$ and $\alpha_k = \beta_k$ for all other k. We say that $\mathfrak{b}$ and $\mathfrak{b}'$ are *chain P-equivalent*, if there exist bilinear Pfister forms $\mathfrak{b}_0, \cdots, \mathfrak{b}_m$ such that $\mathfrak{b}_0 = \mathfrak{b}$, $\mathfrak{b}_m = \mathfrak{b}'$ and every $\mathfrak{b}_i$ for $i \geqslant 1$ is simply P-equivalent to $\mathfrak{b}_{i-1}$.

A *quaternion algebra* over F is a central simple F-algebra of degree 2. Every quaternion algebra Q has a *quaternion basis*, i.e., a basis $\{1, u, v, w\}$ satisfying $u^2 + u \in F$, $v^2 \in F^\times$ and $uv = w = vu + v$ (see [7, p. 25]). It is easily seen that every element $v \in Q \setminus F$ with $v^2 \in F^\times$ extends to a quaternion basis $\{1, u, v, uv\}$ of Q. A tensor product of two quaternion algebras is called a *biquaternion algebra*.

An *involution* on a central simple F-algebra A is an antiautomorphism of A of period 2. Involutions which restrict to the identity on F are said to be of *the first kind*. An involution of the first kind is either *symplectic* or *orthogonal* (see [7, (2.5)]). The *discriminant* of an orthogonal involution σ is denoted by $\operatorname{disc} \sigma$ (see [7, (7.2)]). If K/F is a field extension, the scalar extension of (A, σ) to K is denoted by $(A, \sigma)_K$. We also use the notation $\operatorname{Alt}(A, \sigma) = \{a - \sigma(a) \mid a \in A\}$. According to [7, (2.6 (2))], if σ is of the first kind, the set $\operatorname{Alt}(A, \sigma)$ is an F-linear space of dimension $\frac{1}{2} n(n-1)$, where n is the degree of A.

Let (A, σ) be a totally decomposable algebra of degree 2^n with orthogonal involution over F. In [9], it was shown that there exists a unique, up to isomorphism, subalgebra