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Counting characters above invariant characters in solvable groups

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ABSTRACT

This paper discusses two related questions. First, given a G-invariant character θ of a normal subgroup N of a solvable group, what can we say if the number of characters of G above θ is in some sense as small as possible? Isaacs and Navarro [5] have shown that under certain assumptions about primes dividing the order of the group, one can show that G/N must have a very particular structure. Here we show that these assumptions can be weakened to obtain results about all solvable groups.

We also discuss a related question about blocks. For a prime p and a p-block B of G, we let $k(B)$ denote the number of ordinary characters in B. It is relatively easy to show that $k(B)$ is bounded below by $k(G, D)$, which is the number of conjugacy classes of G that intersect the defect group D of B. In this paper we ask what can be said if equality is achieved. We show that for p-solvable groups, if $k(B) = k(G, D)$, then B is nilpotent and thus $k(B) = |\mathrm{Irr}(D)|$. In addition, we show that this result holds for many blocks of arbitrary finite groups, including all blocks of the symmetric groups. We also extend a result on fully ramified coprime actions in [5].

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1. Introduction

We begin by mentioning a result of Isaacs and Navarro which motivates much of this paper. Recall that if π is a set of primes, we let $k_\pi(G)$ denote the number of conjugacy classes of G of elements of order divisible only by primes in π.

Theorem 1.1. *[5] Let G be a p-solvable group, and if $p = 2$, suppose that $|G|$ is not divisible by a Fermat or Mersenne prime. Let $N \lhd G$ be a p-subgroup, and suppose $\theta \in \mathrm{Irr}(N)$ is G-invariant. If $|\mathrm{Irr}(G|\theta)| = k_{p'}(G/N)$, then G has a normal Sylow p-subgroup.*

In [5] it is shown that Theorem 1.1 is false if G is not assumed to be p-solvable, though it is speculated in [5] that it may be possible to classify the exceptions.

In order to apply this result to blocks, we need to be able to switch the roles of p and p' in the statement of Theorem 1.1 so that we may apply this result to $\mathbf{O}_{p'}(G)$. We also need to remove the hypothesis on Fermat or Mersenne primes. We accomplish both of these here with the following result:

Theorem 1.2. *Let π be a set of primes and let G be a π-separable group. Let $N \lhd G$ be a π-subgroup, and suppose that θ is invariant in G and $|\mathrm{Irr}(G|\theta)| = k_{\pi'}(G/N)$. Then G has a normal Hall π-subgroup.*

Now fix a prime p. Let B be a p-block of G with defect group D, let $k(B)$ denote the number of ordinary irreducible characters in B, and let $\ell(B)$ denote the number of irreducible Brauer characters in B. For a subgroup H of G, we let $k(G, H)$ denote the number of conjugacy classes of G that intersect H. By using results of Brauer about B-elements (see below), it is relatively easy to prove that

$$k(B) \geq k(G, D).$$

We investigate what can be said if equality holds in p-solvable groups.

Theorem 1.3. *Let B be a block of the p-solvable group G with defect group D. If $k(B) = k(G, D)$, then B is nilpotent. In this case we have $\ell(B) = 1$ and $k(G, D) = k(B) = k(D)$.*

We do not know to what extent the hypothesis that G is p-solvable can be removed from Theorem 1.3. However, we are easily able to prove the following:

Theorem 1.4. *Let B be a block of the finite group G with defect group D. Suppose that either D is abelian, or B is the principal block of G. If $k(B) = k(G, D)$, then B is nilpotent, and we have $\ell(B) = 1$ and $k(G, D) = k(B) = k(D)$.*

It is likely that Theorem 1.4 is in the literature, but as our proof is easy and we will need the main idea of the proof elsewhere, we include it here. We will also show, in the