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Perfect and semiperfect restricted enveloping algebras

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ABSTRACT

For a restricted Lie algebra L, the conditions under which its restricted enveloping algebra $u(L)$ is semiperfect are investigated. Moreover, it is proved that $u(L)$ is left (or right) perfect if and only if L is finite-dimensional.

1. Introduction

Let R be a ring with unity and denote by $\mathcal{J}(R)$ the Jacobson radical of R. We recall that R is said to be *semiperfect* if $R/\mathcal{J}(R)$ is Artinian and idempotents of $R/\mathcal{J}(R)$ can

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be lifted to R. Semiperfect rings, introduced by H. Bass in [1], turn out to be a significant class of rings from the viewpoint of homological algebra and representation theory, since they are precisely the rings R for which all finitely generated left or right R-modules have a projective cover (see e.g. [11], Chapter 8, §24). Clearly, one-sided Artinian rings and local rings are semiperfect.

Recall that R is called *left perfect* if all left R-modules have projective covers. Right perfect rings are defined in an analogous way. The pioneering work on perfect rings was carried out by H. Bass in 1960 and most of the main characterizations of these rings are contained in his celebrated paper [1]. In particular, it follows from Bass' results that the following conditions are equivalent: R is left perfect; every flat left R-module is projective; $R/\mathscr{J}(R)$ is Artinian and for every sequence $\{a_i\}$ in $\mathscr{J}(R)$ there exists an integer n such that $a_1a_2\cdots a_n = 0$; R satisfies the descending chain condition on principal right ideals. It should be mentioned that, while semiperfectness is a left–right symmetric property, there exist rings which are perfect on one side but not on the other (see [1], Example 5 on page 476). However, right and left perfectness are clearly equivalent conditions provided R has a nontrivial involution. For instance, this is the case when R is a group algebra or an (ordinary or restricted) enveloping algebra.

Left perfect group rings were characterized by G. Renault in [17] and, independently, by S.M. Woods in [22]. It turns out that, for a group G and a field $\mathbb{F}$, the group algebra $\mathbb{F}G$ is left perfect if and only if G is finite. Subsequently, a generalization of these results to semigroup rings has been carried out by J. Okniński in [14]. On the other hand, although semiperfect group algebras have been also investigated in several papers (see e.g. [3,6,21,23]), a full characterization is not available yet. The best partial result in this direction was obtained by J.M. Goursaud in [6] where, under the assumption that the group G is locally finite, it is proved that $\mathbb{F}G$ is semiperfect if and only if either $\operatorname{char}\mathbb{F} = 0$ and G is finite or $\operatorname{char}\mathbb{F} = p > 0$ and G has a normal p-group of finite index.

In this paper, we consider these problems in the setting of enveloping algebras. For a restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 0$, we denote by $u(L)$ the restricted enveloping algebra of L and by L' the derived subalgebra of L. Recall that a subset S of L is said to be p-nil if every element of S is p-nilpotent. We first provide some necessary and sufficient conditions on L such that $u(L)$ is semiperfect. We show that if $u(L)$ is semiperfect, then every element of L is p-algebraic and L does not contain any infinite-dimensional torus. Under the assumptions that $\mathbb{F}$ is perfect and L' is p-nil we prove that for a locally finite-dimensional restricted Lie algebra L, $u(L)$ is semiperfect if and only if L contains a p-nil restricted ideal of finite codimension. We construct an infinite-dimensional abelian restricted Lie algebra L over an imperfect field K such that $u(L)$ is semiperfect and L has no nonzero p-nil restricted ideal. Hence, the perfectness assumption on the ground field is necessary. It turns out that the structure of semiperfect ordinary enveloping algebras $U(L)$ of arbitrary Lie algebras L is trivial and also quickly discussed.