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Polynomial codimension growth of algebras with involutions and superinvolutions ☆

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ABSTRACT

Let A be an associative algebra over a field F of characteristic zero endowed with a graded involution or a superinvolution $*$ and let $c_n^*(A)$ be its sequence of $*$-codimensions. In [4,12] it was proved that if A is finite dimensional such sequence is polynomially bounded if and only if A generates a variety not containing a finite number of $*$-algebras: the group algebra of $\mathbb{Z}_2$ and a 4-dimensional subalgebra of the 4×4 upper triangular matrices with suitable graded involutions or superinvolutions.

In this paper we focus our attention on such algebras since they are the only finite dimensional $*$-algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety has polynomial growth. We classify the subvarieties of such varieties by giving a complete list of generating finite dimensional $*$-algebras. Along the way we classify all minimal varieties of polynomial growth and surprisingly we show that their number is finite for any given growth. Finally we describe the $*$-algebras whose $*$-codimensions are bounded by a linear function.

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1. Introduction

Let F be a field of characteristic zero and let $F\langle X\rangle$ be the free associative algebra on a countable set X over F. One of the most interesting and challenging problems in combinatorial PI-theory is that of finding numerical invariants allowing to classify the T-ideals of $F\langle X\rangle$, i.e., the ideals invariant under all endomorphisms of $F\langle X\rangle$. There is a well understood connection between T-ideals of $F\langle X\rangle$ and varieties of F-algebras: every T-ideal is the ideal of polynomial identities satisfied by a given variety of algebras. Therefore it is often convenient to translate a given problem on T-ideals into the language of varieties of algebras. A very useful numerical invariant that can be attached to a T-ideal is given by the sequence of codimensions. Such numerical sequence was introduced by Regev in [26] and measures the rate of growth of the multilinear polynomials lying in a given T-ideal. A celebrated theorem of Regev asserts that if A is an associative PI-algebra, i.e., it satisfies a non-trivial polynomial identity, then its sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$, is exponentially bounded. Kemer in [14] proved that for a PI-algebra A, $c_n(A)$ is polynomially bounded if and only if the variety of algebras generated by A does not contain either the Grassmann algebra G of an infinite dimensional vector space or the algebra UT_2 of 2×2 upper triangular matrices. Hence $\mathrm{var}(G)$ and $\mathrm{var}(UT_2)$ are the only varieties of almost polynomial growth, i.e., they grow exponentially but any proper subvariety grows polynomially.

The varieties of polynomial growth were extensively studied in later years (see for instance [5,7,8,16–18]) also in the setting of varieties of graded algebras, algebras with involution, graded involution and superinvolution [4,10–12,27].

In this paper we are interested in the study of associative algebras endowed with a graded involution or a superinvolution. In analogy with the ordinary case, one defines the sequence of $*$-codimensions of a $*$-algebra A, i.e., an algebra endowed with a graded involution or a superinvolution $*$. It turns out that if a $*$-algebra satisfies an ordinary identity, then its sequence of $*$-codimensions is exponentially bounded (see [4,12]). Recently, much interest has been devoted to the study of varieties of $*$-algebras of polynomial growth. More precisely in [4,12] it was proved that a finite dimensional $*$-algebra has polynomial growth of the $*$-codimensions if and only if the corresponding variety does not contain the following algebras: the group algebra of a group of order 2 and a 4-dimensional subalgebra of UT_4, both algebras with suitable graded involutions or superinvolutions. Such algebras are the only finite dimensional $*$-algebras, up to T_2^*-equivalence, generating varieties of almost polynomial growth, i.e., varieties of exponential growth such that any proper subvariety has polynomial growth.

We recall that a variety $\mathcal{V}$ is minimal of polynomial growth if $c_n^*(\mathcal{V}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U} \subsetneq \mathcal{V}$ we have that $c_n^*(\mathcal{U}) \approx q'n^t$ with $t < k$. In this paper we completely classify all subvarieties and all minimal subvarieties of the varieties of almost polynomial growth generated by the above algebras by giving a complete list of finite dimensional $*$-algebras generating them. Moreover we characterize varieties of polynomial growth generated by finite dimensional $*$-algebras by relating