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RANDOM MATRICES: OVERCROWDING ESTIMATES FOR THE SPECTRUM

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ABSTRACT. We address overcrowding estimates for the singular values of random iid matrices, as well as for the eigenvalues of random Wigner matrices. We show evidence of long range separation under arbitrary perturbation even in matrices of discrete entry distributions. In many cases our method yields nearly optimal bounds.

1. INTRODUCTION

1.1. **Random iid matrices with subgaussian tails.** Consider a random matrix $M = (m_{ij})_{1\le i,j\le n}$, where m_{ij} are iid copies of a random variable ξ of mean zero and variance one. Let $\sigma_n \le \cdots \le \sigma_1$ be the singular values of M.

An important problem with practical applications is to bound the condition number of M. As the asymptotic behavior of the largest singular value σ_1 is well understood under natural assumption on ξ, the main problem is to study the lower bound of the least singular value σ_n. This problem was first raised by Goldstine and von Neumann [10] well back in the 1940s, with connection to their investigation of the complexity of inverting a matrix.

To answer Goldstine and von Neumman's question, Edelman [7] computed the distribution of the least singular value of Ginibre matrix (where ξ is standard gaussian). He showed that for all fixed $\varepsilon > 0$

$$\mathbf{P}(\sigma_n \le \varepsilon n^{-1/2}) = \int_0^{\varepsilon^2} \frac{1+\sqrt{x}}{2\sqrt{x}} e^{-(x/2+\sqrt{x})}\, dx + o(1) = \varepsilon - \frac{1}{3}\varepsilon^3 + O(\varepsilon^4) + o(1).$$

Note that the same asymptotic continues to hold for any $\varepsilon > 0$ which can go to zero with n (see also [22])

$$\mathbf{P}(\sigma_n \le \varepsilon n^{-1/2}) \le \varepsilon. \qquad (1)$$

For other singular values of Ginibre matrices, an elegant result by Szarek [24] shows that the σ_{n-k+1} are separated away from zero with an extremely fast rate.

Theorem 1.2. *Assume that ξ is standard gaussian, then there exist absolute constants C_1, C_2 such that for all $\varepsilon > 0$, and all $1 \le k \le n$*

$$(\frac{C_1}{k}\varepsilon)^{k^2} \le \mathbf{P}(\sigma_{n-k+1} \le \frac{\varepsilon}{\sqrt{n}}) \le (\frac{C_2}{k}\varepsilon)^{k^2}.$$

In what follows ε is always bounded by $O(1)$. Motivated by the universality phenomenon in random matrix theory, we expect similar repulsion bounds for general random matrix ensembles. More specifically, we will

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